Modularisation of sequent calculi for normal and non-normal modalities

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In this work we explore the connections between (linear) nested sequent calculi and ordinary sequent calculi for normal and non-normal modal logics. By proposing local versions to ordinary sequent rules we obtain linear nested sequent calculi for a number of logics, including to our knowledge the first nested sequent calculi for a large class of simply dependent multimodal logics, and for many standard non-normal modal logics. The resulting systems are modular and have separate left and right introduction rules for the modalities, which makes them amenable to specification as bipole clauses. While this granulation of the sequent rules introduces more choices for proof search, we show how linear nested sequent calculi can be restricted to blocked derivations, which directly correspond to ordinary sequent derivations.

CCS Concepts: • **Theory of computation** → **Proof theory**; **Modal and temporal logics**; **Linear logic**; *Automated reasoning*;

Additional Key Words and Phrases: Linear nested sequents, labelled systems, modal logic, proof theory

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1 INTRODUCTION

One of the main research topics in proof theory is the proposal of suitable frameworks for logical systems. Determining which properties should be taken into account for calling a framework *suitable* depends on the intended application. For example, *simple* frameworks are easy to understand and handle, hence this can be a desirable characteristic. Another highly desirable property is *analyticity*. Analytic calculi consist solely of rules that compose the formulae to be proved in a stepwise manner. As a result, derivations in an analytic calculus possess the subformula property: every formula that appears (anywhere) in the derivation must be a subformula of the formulae to be proved. This is a powerful restriction on the form of the proofs and can be exploited to prove important meta-logical properties of the formalised logics such as consistency, decidability and interpolation. Also, a framework is often required to be amenable for *smooth extensions*, in order to avoid the necessity of a fresh start every time new axioms are added to the base logic.

Maybe the best known formalism for proposing proof systems is Gentzen's *sequent calculus* [18, 19]. Due to its simplicity, sequent calculus appears as an ideal tool for proving meta-logical properties. However, it is neither expressive enough for constructing analytic calculi for many logics of interest, nor scalable in order to capture large classes of logics in a uniform and systematic way.

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In the case of *modal logics*, the limitation of the sequent framework is glaring. Undoubtedly, there are sequent calculi for a number of modal logics exhibiting many good properties (such as analyticity)[1], which can be used in complexity-optimal decision procedures. However, their construction often seems ad-hoc; they are usually not *modular*, in the sense that the addition of a single property usually implies a reworking of the whole system to obtain cut elimination; and they mostly lack properties such as separate left and right introduction rules for the modalities, which are relevant from the point of view of proof-theoretic semantics and facilitate closer connections to natural deduction systems.

These problems are often connected to the fact that the modal rules in such calculi usually introduce more than one connective at a time. For example, in the standard presentation of the rule

$$\frac{\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Delta}\ \mathsf{k}$$

for modal logic K [10], the context Γ contains an arbitrary finite number of formulae, each of which is prefixed with a box in the conclusion. Thus, if the formulae in Γ really are considered to form part of the context, then this context is not kept intact when passing over to the premiss. Moreover, the context $\Box\Gamma$ in the conclusion places a severe restriction on the side formulae, in that only modalised formulae can appear. Hence the rule is not *local* in the sense that it does not only decompose the principal formula $\Box A$. Alternatively, the k rule can also be seen as an infinite set of rules

$$\left\{ \frac{B_1, \dots, B_n \Rightarrow A}{\Gamma', \Box B_1, \dots, \Box B_n \Rightarrow \Box A, \Delta}\ \mathsf{k}_n \mid n \geq 0 \right\}$$

each with a fixed number of principal formulae. While from this point of view the rules k_n could be considered local because they do not place any restriction on the side formulae in Γ', Δ, they explicitly introduce boxed formulae on both sides of the sequent arrow, and hence explicitly discard the distinction between left and right rules for the modal connective. Thus, both of these perspectives are somewhat dissatisfying. For a more detailed discussion see, e.g., [69].

One way of solving this problem is to consider extensions of the sequent framework that are expressive enough for capturing these modalities using separate left and right introduction rules. This is possible e.g. in the frameworks of *labelled sequents* [17, 54, 65, 68] or in that of *nested sequents* or *tree-hypersequents* [6, 7, 31, 62, 66]. In the labelled sequent framework, the trick is accomplished by explicitly mentioning the Kripke-style relational semantics of normal modal logics in the sequents. In the nested or tree-hypersequent framework in contrast, intuitively, a single sequent is replaced with a tree of sequents, where successors of a sequent are interpreted under a modality. The modal rules of these calculi govern the transfer of (modal) formulae between the different sequents, and it can be shown that it is sufficient to transfer only one formula at a time. However, the price to pay for this added expressivity is that the obvious proof search procedure is of suboptimal complexity since it constructs potentially exponentially large nested sequents [6].

In this work, we reconcile the added superior expressiveness and modularity of nested sequents over ordinary sequents with the computational behaviour of the standard sequent framework by proposing the concept of block form derivations for *linear nested sequents*. Linear nested sequents [37] (short: LNS) is a restricted form of nested sequents where the tree-structure is restricted to that of a line. In LNS, a list of standard sequents is separated by the *nesting operator* $/\!\!/$, with the head of the list interpreted in the usual way as an implication and the tail interpreted (recursively) under a modal operator. The logical rules then act on the elements of the list, possibly moving formulae from one element to another. This finer way of representing systems enables both locality

[1] Analyticity in sequent calculus systems is often guaranteed by proving *cut elimination*.

and modularity by decomposing standard sequent rules into smaller components. For example, the modal rule k in the linear nested setting is decomposed into the two rules

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi\}}{\mathcal{S}\{\Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi\}}\ \Box_L \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \ \Rightarrow A}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \Box A}\ \Box_R$$

Note that different connectives are introduced one at a time by different rules not depending on the formulae in the context, and this entails locality. Moreover, decomposing the sequent rules enables modularity since now extensions of, *e.g.*, the modal system K are obtained by adding the respective (local) modal rules.

However, locality has a collateral side effect: more choices on the application of rules. This may cause an explosion in the proof space. In order to obtain a better control of proofs, we propose a proof strategy based on *blocks of applications* of modal rules. The result is a notion of normal derivations in the linear nested setting, which directly correspond to derivations in the standard sequent setting.

Since we are interested in the connections to the standard sequent framework, we concentrate on logics which have a standard sequent calculus. Examples include normal modal logic K and extensions of it, in particular the family of simply dependent multimodal logics [14], as well as several non-normal modal logics, i.e., standard extensions of *classical modal logic* [10]. Notably, we obtain the first nested sequent calculi for the logics of the *modal tesseract* (see Fig. 12). A prototype implementation of a modular theorem prover using the linear nested sequent calculi is available under https://logic.at/staff/lellmann/lnsprover/.

Finally, while more expressive formalisms such as LNS enable calculi for a broader class of logics, the greater bureaucracy makes it harder to prove meta-logical properties, such as analyticity itself. Since a specific logic gives rise to specific sets of rules in different calculi, it is important to determine whether there is a *general* methodology for determining/analysing such meta-level properties. This is the role of *logical frameworks* in proof theory, where proof systems are adequately embedded into a meta-level formal system so that object-level properties can be uniformly proven. Since logical frameworks often come with automated procedures, the meta-level machinery can be used for proving properties of the embedded systems *automatically*. In [49] *bipoles* and the *focusing proof strategy* [2] in linear logic [21] were used in order to specify sequent systems. By interpreting object-level inference rules as meta-level bipoles, focusing forces a one-to-one correspondence between the application of rules and the derivation of formulae. In this work, we show that this bipole/focusing approach can be extended to linear nested systems. Such specification allows for the proposal of a general theorem prover (POULE available at http://subsell.logic.at/nestLL/), parametric in the theory, profiting from the modularity of the specified systems.

It should be noted that some preliminary results on linear nested systems for various modal systems were presented in [40]. In the present paper we significantly extend these results, give many more examples and refine several technical details. The new contributions with respect to [40] are: (1) generalisation of the results on simply dependent bimodal logics to a large family of logics in Sec. 3.1; (2) introduction of modular linear nested sequent calculi for several logics; in particular, we propose the first local systems for non-normal modal logics of the modal tesseract in Sec. 4; (3) definition of a notion of normal forms for linear nested sequents, via the concept of *modal block forms*; this allows for a modular way of translating modal sequents into linear nested sequent systems; (4) automatic generation of labelled systems for all the logics in the modal tesseract; and finally (5) discussion on some other possible approaches for focusing in modal systems, especially the ones proposed in [8].

The rest of the paper is organized as follows. In Section 2 we introduce the concept of linear nested sequents (LNS). In Section 3 we show that the linear nested sequent framework is a good formalism for a large class of modal systems, showing non trivial extensions of multimodal K as well as a large class of non-normal modal logics. Section 4 also presents local systems for non-normal logics, but by modifying the structural rules of the system, instead of their logical rules. In both Sections we make use of auxiliary structural operators. Since locality often entails less efficient systems, in Section 5 we propose a notion of "normal proofs" in LNS derivations, hence showing how to reduce the proof space and consequently optimize proof search. Since modal connectives presented in this work are uniquely defined by the modal rules, we can specify such rules as bipoles. We show the specification process in Section 6, by first proposing labelled sequent versions for LNS systems and then showing how to to generate bipole clauses in linear logic which adequately correspond to LNS modal rules. Finally, in Section 7, we conclude by outlining some future work.

In the remainder of this article we assume familiarity with some basic notions of modal logic. See, e.g., [5, 10, 33] for an introduction.

2 LINEAR NESTED SEQUENT SYSTEMS

As an intermediate between the efficiency of the ordinary sequent framework and the expressiveness of the nested sequent framework [6, 7, 31, 62, 66] we consider calculi in the *linear nested sequent* framework [37]. This is essentially a reformulation of Masini's 2-sequents [43] in the nested sequent framework, where the tree structure of nested sequents is restricted to that of a line. The benefit is that this framework exhibits the structure essential to obtain modular calculi, i.e., the nesting of sequents, while retaining a very close connection to the ordinary sequent framework and offering advantages in terms of efficiency. A similar approach was followed with the G-CK$_n$ sequents for constructive modal logic of [46] which moreover also add some form of focusing to the linear structure. The superior expressiveness of the linear nested sequent compared to the ordinary sequent framework is witnessed, e.g., by analytic calculi for temporal logics [29, 30] or intermediate logic LC [34], for which logics no analytic ordinary sequent calculi seem to exist.

In the following, we consider a *sequent* to be a pair $\Gamma \Rightarrow \Delta$ of multisets of formulae and adopt the standard conventions and notations for formulae, multisets, and proof systems (see e.g. [67]). Note that the sequents are two-sided, even though the underlying propositional logic for all the considered logics is classical. The benefit of this formulation over one-sided sequents in the style of [6] is that it avoids the introduction of dual modal operators. A linear nested sequent then is simply a finite list of sequents. As noted in [37], this data structure matches exactly that of a *history* in a backwards proof search in an ordinary sequent calculus, a fact we will use heavily.

Definition 2.1. The set LNS of *linear nested sequents* is given recursively by:

(1) if $\Gamma \Rightarrow \Delta$ is a sequent then $\Gamma \Rightarrow \Delta \in \mathsf{LNS}$
(2) if $\Gamma \Rightarrow \Delta$ is a sequent and $\mathcal{G} \in \mathsf{LNS}$ then $\Gamma \Rightarrow \Delta \mathbin{/\!\!/} \mathcal{G} \in \mathsf{LNS}$.

We will write $\mathcal{S}\{\Gamma \Rightarrow \Delta\}$ for denoting a *context* $\mathcal{G} \mathbin{/\!\!/} \Gamma \Rightarrow \Delta \mathbin{/\!\!/} \mathcal{H}$ where each of $\mathcal{G}, \mathcal{H}$ is a linear nested sequent or empty (omitting the $/\!\!/$ symbol in the latter case). We call each sequent in a linear nested sequent a *component* and slightly abuse notation, abbreviating "linear nested sequent" to LNS. The standard interpretation for linear nested sequents for modal logic K is given by:

$$\iota_\Box(\Gamma \Rightarrow \Delta) := \bigwedge \Gamma \to \bigvee \Delta$$
$$\iota_\Box(\Gamma \Rightarrow \Delta \mathbin{/\!\!/} \mathcal{G}) := \bigwedge \Gamma \to \bigvee \Delta \vee \Box\iota_\Box(\mathcal{G})$$

As usual, we take a conjunction and disjunction over an empty multiset to be $\top$ and $\bot$, respectively.

Thus, the nesting operator $/\!\!/$ of linear nested sequents is interpreted as a structural connective for the modal box on the right hand side of a sequent. Note that this is essentially the standard interpretation of the brackets [.] of nested sequents using the two-sided sequents of [7] instead of the single-sided formulation of [6]. Since we only consider *linear* nested sequents, we use $/\!\!/$ instead of iterated brackets to increase readability.

Example 2.2. Consider the logic K.

(1) The formula interpretation of the linear nested sequent $\Rightarrow A /\!\!/ A \Rightarrow$ is $\top \to A \lor \Box(A \to \bot)$ which is equivalent to $A \lor \Box\neg A$.
(2) The formula interpretation of the linear nested sequent $\Box A \Rightarrow /\!\!/ \Rightarrow /\!\!/ \Rightarrow A$ is $\Box A \to \bot \lor \Box(\top \to \bot \lor \Box(\top \to A))$ which is equivalent to $\Box A \to \Box\Box A$.

Remark 1. *It is worth noting that while the structure of a linear nested sequent as a list of ordinary sequents is the same as that of a* hypersequent *(see, e.g., [4]), there is an important difference between the two frameworks. In virtually all hypersequent calculi the formula interpretation of a hypersequent is given by some form of disjunction. E.g., in the context of modal logics the standard formula interpretation of the hypersequent* $\Gamma_1 \Rightarrow \Delta_1 \mid \Gamma_2 \Rightarrow \Delta_2 \mid \Gamma_3 \Rightarrow \Delta_3$ *would be given by* $\Box(\bigwedge \Gamma_1 \to \bigvee \Delta_1) \lor \Box(\bigwedge \Gamma_2 \to \bigvee \Delta_2) \lor \Box(\bigwedge \Gamma_3 \to \bigvee \Delta_3)$. *In particular, every component of the hypersequent is interpreted uniformly under exactly one application of* $\Box$. *In contrast, the formula interpretation of the linear nested sequent* $\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \Gamma_2 \Rightarrow \Delta_2 /\!\!/ \Gamma_3 \Rightarrow \Delta_3$ *according to the interpretation* $\iota_\Box$ *from above is given by* $\bigwedge \Gamma_1 \to \bigvee \Delta_1 \lor \Box(\bigwedge \Gamma_2 \to \bigvee \Delta_2 \lor \Box(\bigwedge \Gamma_3 \to \bigvee \Delta_3))$. *Crucially, every component is interpreted under a number of modal operators which depends on its position in the linear nested sequent. So the formula interpretations of the hypersequents* $\Rightarrow A \mid A \Rightarrow$ *and* $\Box A \Rightarrow \mid \Rightarrow \mid \Rightarrow A$ *corresponding to the linear nested sequents of Ex. 2.2 would be given by the formulae* $\Box(\top \to A) \lor \Box(A \to \bot)$ *and* $\Box(\Box A \to \bot) \lor \Box(\top \to \bot) \lor \Box(\top \to A)$ *respectively, which are equivalent to the formulae* $\Box A \lor \Box\neg A$ *and* $\Box\neg\Box A \lor \Box\bot \lor \Box A$ *respectively. Clearly, these interpretations are rather different from the ones in the linear nested sequent framework. Accordingly, virtually all hypersequent calculi contain a rule like the* external exchange rule *which permits a reordering of the components. However, under the linear nested sequent formula interpretation and for the logics considered here this rule would not be sound. In line with this observation, linear nested sequent calculi have also been considered as hypersequent calculi without the external exchange rule under the name of* non-commutative hypersequents *e.g. in [29, 30], in a more semantically focused version in [59], and in their tableaux version as* path-hypertableaux *in [11]. A more detailed investigation on the connection between linear nested sequents and hypersequents is contained in [37].*

In this work we consider only modal logics based on classical propositional logic, and we take the system $\mathsf{LNS_G}$ (Fig. 1) as our base calculus. The linear nested sequent versions of the standard (internal) structural rules are given in Fig. 2.

Definition 2.3. For a system C of linear nested sequent rules, we define a *derivation* to be a finite directed tree where each node is labelled with a linear nested sequent in such a way that the linear nested sequent associated to each node is obtained from the linear nested sequents associated to its immediate successors by an application of one of the rules from C. In particular, each leaf of a derivation is labelled with the conclusion of an instance of a zero-premiss rule, i.e., one of the rules init, $\bot_L$, $\top_R$. The label of the root of a derivation is called the *conclusion* of that derivation, and we say that a linear nested sequent $\mathcal{G}$ is *derivable in the system* C, in symbols $\vdash_C \mathcal{G}$, if there is a derivation in C with conclusion $\mathcal{G}$. The *depth* of a derivation is the length of the longest branch in the underlying directed tree plus one. In the following we will denote by $\mathsf{LNS}_\mathcal{L}$ a linear nested sequent system for a logic $\mathcal{L}$ obtained by adding a certain set of rules for the modal operators to the system $\mathsf{LNS_G}$. By $\mathsf{LNS}_\mathcal{L}\mathsf{ConW}$ we denote the extension of the system $\mathsf{LNS}_\mathcal{L}$ with the structural rules of contraction and weakening from Fig. 2, abbreviating $\mathsf{C}_L, \mathsf{C}_R$ to Con and $\mathsf{W}_L, \mathsf{W}_R$ to W.

Observe that $\mathsf{LNS_G}$ is the linear nested version of the well known system G3cp from [67] plus explicit rules for negation. The reason for considering the structural rules explicitly is that, while

$$\frac{}{\mathcal{S}\{\Gamma, p \Rightarrow p, \Delta\}}\ \text{init} \quad \frac{}{\mathcal{S}\{\Gamma, \bot \Rightarrow \Delta\}}\ \bot_L \quad \frac{}{\mathcal{S}\{\Gamma \Rightarrow \top, \Delta\}}\ \top_R \quad \frac{\mathcal{S}\{\Gamma \Rightarrow A, \Delta\}}{\mathcal{S}\{\Gamma, \neg A \Rightarrow \Delta\}}\ \neg_L \quad \frac{\mathcal{S}\{\Gamma, A \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma \Rightarrow \neg A, \Delta\}}\ \neg_R$$

$$\frac{\mathcal{S}\{\Gamma, A \Rightarrow \Delta\} \quad \mathcal{S}\{\Gamma, B \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, A \vee B \Rightarrow \Delta\}}\ \vee_L \quad \frac{\mathcal{S}\{\Gamma, A, B \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, A \wedge B \Rightarrow \Delta\}}\ \wedge_L \quad \frac{\mathcal{S}\{\Gamma, B \Rightarrow \Delta\} \quad \mathcal{S}\{\Gamma \Rightarrow A, \Delta\}}{\mathcal{S}\{\Gamma, A \to B \Rightarrow \Delta\}}\ \to_L$$

$$\frac{\mathcal{S}\{\Gamma \Rightarrow A, B, \Delta\}}{\mathcal{S}\{\Gamma \Rightarrow A \vee B, \Delta\}}\ \vee_R \quad \frac{\mathcal{S}\{\Gamma \Rightarrow A, \Delta\} \quad \mathcal{S}\{\Gamma \Rightarrow B, \Delta\}}{\mathcal{S}\{\Gamma \Rightarrow A \wedge B, \Delta\}}\ \wedge_R \quad \frac{\mathcal{S}\{\Gamma, A \Rightarrow B, \Delta\}}{\mathcal{S}\{\Gamma \Rightarrow A \to B, \Delta\}}\ \to_R$$

Fig. 1. System $\mathsf{LNS_G}$ for classical propositional logic. In the init rule, p is atomic.

$$\frac{\mathcal{S}\{\Gamma, A, A \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, A \Rightarrow \Delta\}}\ \mathsf{C}_L \quad \frac{\mathcal{S}\{\Gamma \Rightarrow A, A, \Delta\}}{\mathcal{S}\{\Gamma \Rightarrow A, \Delta\}}\ \mathsf{C}_R \quad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, A \Rightarrow \Delta\}}\ \mathsf{W}_L \quad \frac{\mathcal{S}\{\Gamma \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma \Rightarrow A, \Delta\}}\ \mathsf{W}_R$$

Fig. 2. The structural rules of contraction and weakening.

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta /\!\!/ \Sigma, A \Rightarrow \Pi\}}{\mathcal{S}\{\Gamma, \Box A \Rightarrow \Delta /\!\!/ \Sigma \Rightarrow \Pi\}}\ \Box_L \qquad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \ \Rightarrow A}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta, \Box A}\ \Box_R$$

Fig. 3. The modal rules of the linear nested sequent calculus $\mathsf{LNS_K}$ for K.

in the logical systems considered in Section 3.2 contraction and weakening are admissible (see Lemmas 3.18 and 3.22), some of the systems in Sections 3.1 and 4 are based on sequent calculi which include explicit contraction and weakening. As a side remark, it is worth noticing that the approach presented here could be easily adapted to having LKF [41] as the base logical system, since such a decision would not alter the proof theory developed for the modal connectives.

Fig. 3 presents the modal rules for the linear nested sequent calculus $\mathsf{LNS_K}$ for K, essentially a linear version of the standard nested sequent calculus from [6, 62]. Thus, the calculus $\mathsf{LNS_K}$ contains the rules of $\mathsf{LNS_G}$ together with the rules of Fig. 3.

Conceptually, the main point is that the sequent rule k is split into the two rules $\Box_L$ and $\Box_R$, which permit to simulate the sequent rule treating one formula at a time. While this is one of the main features of nested sequent calculi and deep inference in general [25], being able to separate the left/right behaviour of the modal connectives is the key to modularity for nested and linear nested sequent calculi [37, 66]. It is worth noting that the same phenomenon was also previously observed in the dual setting of *prefixed tableaux* in [44]. There, the prefixed tableaux versions of a number of standard nested sequent rules under the correspondence given in [15] were constructed by decomposing a large jump along the accessibility relation "into many "leaner" steps" dealing with only one formula each (*ib.*, p.329). In addition to the decomposition of ordinary sequent/tableaux rules in these works, here we moreover restrict the tree structure of the nested sequents/prefixed tableaux to that of a line.

Completeness of $\mathsf{LNS_K}$ w.r.t. modal logic K is shown by simulating a sequent derivation bottom-up in the last two components of the linear nested sequents, marking applications of modal rules by the nesting $/\!\!/$ and simulating the k-rule by a block of $\Box_R$ and $\Box_L$ rules [37]. Hence, an application of k on a branch with history captured by the LNS $\mathcal{G}$ is simulated by:

$$\cfrac{\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Delta} \, \mathsf{k} \quad \vdots\, \mathcal{G} \qquad \rightsquigarrow \qquad \cfrac{\cfrac{\mathcal{G} /\!\!/ \Gamma' \Rightarrow \Delta /\!\!/ \Gamma \Rightarrow A}{\mathcal{G} /\!\!/ \Gamma', \Box\Gamma \Rightarrow \Delta /\!\!/ \Rightarrow A}\, \Box_L}{\mathcal{G} /\!\!/ \Gamma', \Box\Gamma \Rightarrow \Box A, \Delta}\, \Box_R$$

where the double line indicates multiple rule applications. Observe that this method relies on the view of linear nested sequents as histories in proof search, where intuitively the modal rules mark a transition or jump to a new state in a corresponding Kripke model, compare, e.g., [22] for a discussion in the dual framework of tableaux systems. It also simulates the propositional sequent rules in the *rightmost* component of the linear nested sequents. However, while the principal formulae of the sequent rule can now be handled separately, the modal rules in the LNS system do not need to occur in a block corresponding to one application of the sequent rule anymore. In fact, one way of deriving the instance $\Box(p \to q) \to (\Box p \to \Box q)$ of the normality axiom for modal logic K is as follows.

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{}{\Box p \Rightarrow /\!\!/ q \Rightarrow q}\,\mathsf{init} \quad \cfrac{\cfrac{}{\Rightarrow /\!\!/ p \Rightarrow p, q}\,\mathsf{init}}{\Box p \Rightarrow /\!\!/ \Rightarrow p, q}\,\Box_L}{\Box p \Rightarrow /\!\!/ p \to q \Rightarrow q}\,\to_L}{\Box(p \to q), \Box p \Rightarrow /\!\!/ \Rightarrow q}\,\Box_L}{\Box(p \to q), \Box p \Rightarrow \Box q}\,\Box_R}{\Rightarrow \Box(p \to q) \to (\Box p \to \Box q)}\,\to_R$$

Note that the propositional rule $\to_L$ is applied between two modal rules. Hence there are many derivations in $\mathsf{LNS_K}$ which are not the result of simulating a derivation of the sequent calculus for K. Thus, while the linear nested sequent calculus $\mathsf{LNS_K}$ has conceptual advantages over the standard sequent calculus for K, there are many more possible derivations with the same conclusion, when compared to the sequent calculus. In Section 6, we will consider how to restrict proof search to a smaller class of derivations, while retaining the conceptual advantages of the framework.

3 LINEAR NESTED SEQUENT SYSTEMS AND MODALITY

In [37] the method of granulising sequent rules into linear nested sequent rules was applied to some basic modal logics and to the multi-succedent calculus for intuitionistic logic. In the following, we will considerably extend these results and show that the linear nested sequent framework is a *good* formalism for a large class of modal systems. We first concentrate on non trivial extensions of multimodal K (the so called simply dependent multimodal logics) and then we show how to modularly extend the LNS approach also for handling non-normal modal logics.

3.1 Simply dependent multimodal logics

As a first example we consider multimodal logics with a simple interaction between the modalities, called *simply dependent multimodal logics* [14]. The language for these logics contains indexed modalities $\Box_i$ for indices i from an index set $N \subseteq \mathbb{N}$ of natural numbers. In a Hilbert-style presentation of these logics, the axioms are given by extensions of the axioms of modal logic K for every modality $\Box_i$ together with *interaction axioms* of the form $\Box_i A \to \Box_j A$. A simple example of such a logic is the simply dependent bimodal logic $\mathsf{KT} \oplus_\subseteq \mathsf{S4}$ from [14], whose language contains the two modalities $\Box_1$ and $\Box_2$. Its Hilbert-style axiomatisation is given by the axioms and rules of classical propositional logic together with the axioms and rules of modal logic KT for the modality $\Box_1$, i.e., axioms K and T and the rule nec, the S4 axioms for the modality $\Box_2$, i.e., axioms K, T, 4 and rule nec, and the single interaction axiom $\Box_2 A \to \Box_1 A$. As usual, the rules of modus ponens and nec can be applied to any previously derived theorem. Standard modal logics such as K or extensions

K $\Box(A \to B) \to (\Box A \to \Box B)$ $\quad \dfrac{A}{\Box A}$ nec $\quad$ D $\neg(\Box A \wedge \Box\neg A)$ $\quad$ T $\Box A \to A$ $\quad$ 4 $\Box A \to \Box\Box A$

Fig. 4. Some modal axioms and rule nec. Modal logic K contains the propositional tautologies, modus ponens, K and nec.

also can be seen as the trivial case of simply dependent multimodal logics where the index set N is a singleton. Other examples include multimodal logics with a *justified knowledge* or "any fool knows" modality from [3, 45], or even substructural logics with *subexponentials* [13]. Here and in the following we will identify a logic with its set of theorems and write $A \in \mathcal{L}$ if the formula A is a theorem of logic $\mathcal{L}$, i.e., derivable in the Hilbert-style system for $\mathcal{L}$.

A general framework to describe simply dependent multimodal logics was given in [1, Sec. 4]. There, such a logic is given essentially by a triple $(N, \preccurlyeq, F)$, where N is a finite set of natural numbers, $(N, \preccurlyeq)$ is a partial order (i.e., transitive, reflexive and antisymmetric), and F is a mapping from N to a set $\mathfrak{L}$ of logics.

In the present work, we will take $\mathfrak{L}$ to be the set of extensions of modal logic K with axioms from the set $\{\mathsf{D}, \mathsf{T}, 4\}$ (see Fig. 4). The *logic described by* $(N, \preccurlyeq, F)$ then has modalities $\Box_i$ for every $i \in N$, with axioms for the modality i given by the logic $F(i)$ and interaction axioms $\Box_j A \to \Box_i A$ for every $i, j \in N$ with $i \preccurlyeq j$. We write $\mathcal{L}_{(N,\preccurlyeq,F)}$ for the logic described by $(N, \preccurlyeq, F)$.

Example 3.1. The simply dependent bimodal logic $\mathsf{KT} \oplus_{\subseteq} \mathsf{S4}$ is given by the description $(N, \preccurlyeq, F)$ with $N = \{1, 2\}$, and $F(1) = \mathsf{KT}$, $F(2) = \mathsf{S4}$, where $\preccurlyeq$ is given by $1 \preccurlyeq 1, 1 \preccurlyeq 2, 2 \preccurlyeq 2$.

The following definition extends the concept of frames to simply dependent multimodal logic. The notions of valuations, model and truth in a world of the model are defined as usual (see, e.g., [5, 10]). As usual, we also identify a logic with the set of its theorems and write $\mathcal{L}_1 \subseteq \mathcal{L}_2$ if every theorem of $\mathcal{L}_1$ is also a theorem of $\mathcal{L}_2$.

Definition 3.2. Let $(N, \preccurlyeq, F)$ be a description for a simply dependent multimodal logic. A $(N, \preccurlyeq, F)$*-frame* is a tuple $(W, (R_i)_{i \in N})$ consisting of a set W of *worlds* and an *accessibility relation* R_i for every index $i \in N$, such that for all $i, j \in N$:

- If $\mathsf{KD} \subseteq F(i)$, then R_i is serial.
- If $\mathsf{KT} \subseteq F(i)$, then R_i is reflexive.
- If $\mathsf{K4} \subseteq F(i)$, then R_i is transitive.
- If $i \preccurlyeq j$, then $R_i \subseteq R_j$.

Since here we only consider simply dependent multimodal logics where the different component logics are extensions of K with axioms from $\{\mathsf{D}, \mathsf{T}, 4\}$, and since the interaction axioms are of a particularly simple shape, standard results e.g. from Sahlqvist theory [5, Thm. 4.42] immediately yield soundness and completeness:

THEOREM 3.3. *The modal logic given by the description $(N, \preccurlyeq, F)$ is the logic of the class of $(N, \preccurlyeq, F)$-frames, i.e., a formula is a theorem of the logic $\mathcal{L}_{(N,\preccurlyeq,F)}$ iff it is valid in all $(N, \preccurlyeq, F)$-frames.*

By standard modal reasoning we immediately obtain the following lemma stating upwards propagation of the modal axioms D and T.

LEMMA 3.4. *Let $(N, \preccurlyeq, F)$ be a description of a simply dependent multimodal logic $\mathcal{L}$. Then for every $i \in N$:*

- *If $\mathsf{KD} \subseteq F(i)$, then for every $j \in N$ with $i \preccurlyeq j$, $\neg(\Box_j A \wedge \Box_j \neg A)$ is also a theorem of $\mathcal{L}$.*
- *If $\mathsf{KT} \subseteq F(i)$, then for every $j \in N$ with $i \preccurlyeq j$, $\Box_j A \to A$ is also a theorem of $\mathcal{L}$.*

$$\frac{\{\Box_j\Gamma_j, \Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^{\neg 4}(i)\} \Rightarrow A}{\Omega, \{\Box_j\Gamma_j, \Box_j\Sigma_j : j \in \uparrow^4(i)\}, \{\Box_j\Sigma_j : j \in \uparrow^{\neg 4}(i)\} \Rightarrow \Box_i A, \Xi}\ \mathsf{k}_i$$

$$\frac{\{\Box_j\Gamma_j, \Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^{\neg 4}(i)\} \Rightarrow}{\Omega, \{\Box_j\Gamma_j, \Box_j\Sigma_j : j \in \uparrow^4(i)\}, \{\Box_j\Sigma_j : j \in \uparrow^{\neg 4}(i)\} \Rightarrow \Xi}\ \mathsf{d}_i \qquad \frac{\Omega, \{\Sigma_j : j \in \uparrow(i)\} \Rightarrow \Xi}{\Omega, \{\Box_j\Sigma_j : j \in \uparrow(i)\} \Rightarrow \Xi}\ \mathsf{t}_i$$

$$\mathcal{R}_{(N,\preccurlyeq,F)} := \{\mathsf{k}_i : i \in N\} \cup \{\mathsf{d}_i : i \in N, \mathsf{KD} \subseteq F(i)\} \cup \{\mathsf{t}_i : i \in N, \mathsf{KT} \subseteq F(i)\}$$

Fig. 5. The modal sequent rules for the simply dependent multimodal logic given by the transitive-closed description $(N, \preccurlyeq, F)$.

Proof. By closing the axioms under the interaction axioms $\Box_j A \to \Box_i A$ for $i \preccurlyeq j$. Alternatively, this can be seen from the semantical characterisation. □

Hence we may assume, without loss of generality, that for any description $(N, \preccurlyeq, F)$ and any $i \in N$, if $\mathsf{KD} \subseteq F(i)$ (or $\mathsf{KT} \subseteq F(i)$), then for every $j \in N$ with $i \preccurlyeq j$ we have $\mathsf{KD} \subseteq F(j)$ (resp. $\mathsf{KT} \subseteq F(j)$). As a more economic notation we also write $\uparrow(i)$ for the *upset* of the index i, i.e., the set $\{j \in N : i \preccurlyeq j\}$. Furthermore, in light of the comments above we extend this notation to the sets $\uparrow^{\mathsf{Ax}}(i) := \{j \in N : i \preccurlyeq j,\ \mathsf{KAx} \subseteq F(j)\}$ and $\uparrow^{\neg\mathsf{Ax}}(i) := \{j \in N : i \preccurlyeq j,\ \mathsf{KAx} \nsubseteq F(j)\}$ where Ax is any of the axioms D, T, 4. Thus e.g. the set $\uparrow^{\neg 4}(i)$ is the set of indices j with $i \preccurlyeq j$ such that $\mathsf{K4} \nsubseteq F(j)$, i.e., the logic $F(j)$ does not derive the transitivity axiom 4.

The next step is to obtain cut-free sequent calculi for logics of this family. In order to obtain cut-free completeness, i.e., completeness without the cut rule, we also need the set of transitive logics to be upwards closed. Formally:

Definition 3.5. A description $(N, \preccurlyeq, F)$ is *transitive-closed* if for every $i, j \in N$ with $i \preccurlyeq j$, if $\mathsf{K4} \subseteq F(i)$ then $\mathsf{K4} \subseteq F(j)$.

Using the method of *cut elimination by saturation* for sequent rules with restrictions on the context, as developed in [36, 39], it is then reasonably straightforward to construct cut-free sequent calculi for simply dependent multimodal logics given by a transitive-closed description. Since the actual construction of the sequent rules is not central to this paper, we will omit the details. The resulting modal rules and rule sets are given in Fig. 5.

Definition 3.6. The restriction of the propositional calculus $\mathsf{LNS_G}$ from Fig. 1 to sequents is denoted by G. If $(N, \preccurlyeq, F)$ is a description for a simply dependent multimodal logic, then $\mathrm{G}_{(N,\preccurlyeq,F)}$ is the sequent calculus extending the propositional calculus G with the modal rules $\mathcal{R}_{(N,\preccurlyeq,F)}$ according to Fig. 5.

The intuition behind the rules perhaps is best obtained by considering an example:

Example 3.7. Continuing our Ex. 3.1, in the case of the logic $\mathsf{KT} \oplus_{\subseteq} \mathsf{S4}$ we have $\mathsf{KD} \subseteq \mathsf{KT} \subseteq F(i)$ for $i = 1, 2$ and $\mathsf{K4} \subseteq F(2)$ but $\mathsf{K4} \nsubseteq F(1)$. Hence we have $\uparrow(1) = \{1, 2\}, \uparrow(2) = \{2\}$ and furthermore $\uparrow^4(1) = \{2\}, \uparrow^{\neg 4}(1) = \{1\}$ as well as $\uparrow^4(2) = \{2\}, \uparrow^{\neg 4}(2) = \emptyset$. Thus the sequent calculus $\mathrm{G}_{\mathsf{KT}\oplus_{\subseteq}\mathsf{S4}}$ for this logic contains the following modal rules, obtained as specific instances of the rules given in Fig. 5:

$$\frac{\Box_2\Gamma_2, \Sigma_2, \Sigma_1 \Rightarrow A}{\Omega, \Box_2\Gamma_2, \Box_2\Sigma_2, \Box_1\Sigma_1 \Rightarrow \Box_1 A, \Xi}\ \mathsf{k}_1 \qquad \frac{\Box_2\Gamma_2, \Sigma_2 \Rightarrow A}{\Omega, \Box_2\Gamma_2, \Box_2\Sigma_2 \Rightarrow \Box_2 A, \Xi}\ \mathsf{k}_2$$

$$\frac{\Box_2\Gamma_2, \Sigma_2, \Sigma_1 \Rightarrow}{\Omega, \Box_2\Gamma_2, \Box_2\Sigma_2, \Box_1\Sigma_1 \Rightarrow \Theta}\ \mathsf{d}_1 \qquad \frac{\Box_2\Gamma_2, \Sigma_2 \Rightarrow}{\Omega, \Box_2\Gamma_2, \Box_2\Sigma_2 \Rightarrow \Theta}\ \mathsf{d}_2 \qquad \frac{\Omega, \Sigma_1 \Rightarrow \Theta}{\Omega, \Box_1\Sigma_1 \Rightarrow \Theta}\ \mathsf{t}_1 \qquad \frac{\Omega, \Sigma_2 \Rightarrow \Theta}{\Omega, \Box_2\Sigma_2 \Rightarrow \Theta}\ \mathsf{t}_2$$

Note that this rule set could still be simplified in two ways. First we observe that the rules $\mathsf{d}_1, \mathsf{d}_2$ are derivable using rules $\mathsf{t}_1, \mathsf{t}_2$ and weakening, and hence could be omitted from the rule set. For the sake of a uniform presentation we decided to keep them. Further, the rules t_i could be restricted to the more traditional version with only a single principal formula. We chose the current formulation since this is the form of the rules which is obtained directly from the construction and facilitates a more uniform cut elimination proof.

Remark 2. *The previous example also serves to illustrate the issues with* modularity *in the sequent framework: suppose we wanted to obtain a cut-free sequent system for the logic* $\mathsf{KT} \oplus_{\subseteq} \mathsf{KT}$ *instead of the logic* $\mathsf{KT} \oplus_{\subseteq} \mathsf{S4}$*, i.e., we only drop the axiom* $\Box_2 A \to \Box_2\Box_2 A$ *from the Hilbert-style system. Then to obtain the calculus* $\mathsf{G}_{\mathsf{KT}\oplus_{\subseteq}\mathsf{KT}}$ *from the calculus* $\mathsf{G}_{\mathsf{KT}\oplus_{\subseteq}\mathsf{S4}}$ *above we would need to drop the context formulae* $\Box_2\Gamma_2$ *from* each *of the rules* $\mathsf{k}_1, \mathsf{k}_2, \mathsf{d}_1, \mathsf{d}_2$*. This is no accident: in general, adding or deleting one axiom from the Hilbert-style presentation of a logic requires heavy modifications of the corresponding sequent calculi, which need to take* all *rules of that calculus into account.*

While soundness and completeness of the calculi $\mathsf{G}_{(N,\preccurlyeq,F)}\mathsf{ConW}$ follow directly from the construction, for later reference and the reader not familiar with the general construction we state them explicitly and briefly sketch the proofs.

Theorem 3.8. *Let* $(N, \preccurlyeq, F)$ *be a transitive-closed description of a simply dependent multimodal logic. Then the sequent calculus* $\mathsf{G}_{(N,\preccurlyeq,F)}\mathsf{ConW}$ *is sound with respect to this logic, i.e., for every formula A we have that* $A \in \mathcal{L}_{(N,\preccurlyeq,F)}$ *if* $\vdash_{\mathsf{G}_{(N,\preccurlyeq,F)}\mathsf{ConW}} \Rightarrow A$.

Proof. We use the fact that the logic given by the description is also characterised by frames $(W, (R_i)_{i\in N})$ where for $i \in N$ the accessibility relation R_i satisfies the properties stipulated by the logic $F(i)$ (i.e., is serial if $\mathsf{KD} \subseteq F(i)$, reflexive if $\mathsf{KT} \subseteq F(i)$ and transitive if $\mathsf{K4} \subseteq F(i)$), and where for every $i, j \in N$ with $i \preccurlyeq j$ we have $R_i \subseteq R_j$. Then it is easy to show that all the modal rules preserve validity by showing that if the negation of the conclusion is satisfiable in such a frame, then so is the premiss. Since the zero-premiss rules are valid, i.e., the negation of their formula interpretation is not satisfiable in any frame, from this we obtain the soundness statement by induction on the depth of the derivation. As an example we fix a description $(N, \preccurlyeq, F)$ and consider the following application of the rule d_i for an index $i \in N$ such that $F(i)$ is serial.

$$\frac{\{\Box_j\Gamma_j, \Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^{\neg 4}(i)\} \Rightarrow}{\Omega, \{\Box_j\Gamma_j, \Box_j\Sigma_j : j \in \uparrow^4(i)\}, \{\Box_j\Sigma_j : j \in \uparrow^{\neg 4}(i)\} \Rightarrow \Xi}\ \mathsf{d}_i$$

If the negation of the conclusion of this rule is satisfiable in a $(N, \preccurlyeq, F)$-model $\mathfrak{M} = (W, (R_i)_{i\in N}, \sigma)$, then we have a world $w \in W$ such that

$$\mathfrak{M}, w \Vdash \bigwedge \Omega \wedge \bigwedge_{j\in\uparrow^4(i)} \Big(\bigwedge \Box_j\Gamma_j \wedge \bigwedge \Box_j\Sigma_j\Big) \wedge \bigwedge_{j\in\uparrow^{\neg 4}(i)} \bigwedge \Box_j\Sigma_j \wedge \neg \bigvee \Xi\ . \tag{1}$$

Since $F(i)$ is serial, there is a world $v \in W$ with wR_iv, and since $i \preccurlyeq j$ implies $R_i \subseteq R_j$ for all $i, j \in N$, for this v we also have wR_jv for every j with $i \preccurlyeq j$. Hence using (1) and transitivity of the relations R_j for j with $\mathsf{K4} \subseteq F(j)$ we obtain

$$\mathfrak{M}, v \Vdash \bigwedge_{j\in\uparrow^4(i)} \Big(\bigwedge \Box_j\Gamma_j \wedge \bigwedge \Sigma_j\Big) \wedge \bigwedge_{j\in\uparrow^{\neg 4}(i)} \bigwedge \Sigma_j\ .$$

Hence the negation of the interpretation of the premiss of this rule application is satisfied in v. The reasoning for the remaining rules is similar. ◻

Theorem 3.9. *Let $(N, \preccurlyeq, F)$ be a transitive-closed description of a simply dependent multimodal logic. Then the sequent calculus $\mathsf{G}_{(N,\preccurlyeq,F)}\mathsf{ConW}$ is (cut-free) complete with respect to this logic, i.e., for every formula A we have that $A \in \mathcal{L}_{(N,\preccurlyeq,F)}$ only if $\vdash_{\mathsf{G}_{(N,\preccurlyeq,F)}\mathsf{ConW}} \Rightarrow A$.*

Proof. As usual, completeness of the system without a cut rule is shown by first showing that every axiom and rule of the Hilbert-style system for the logic $\mathcal{L}_{(N,\preccurlyeq,F)}$ can be simulated in the system $\mathsf{G}_{(N,\preccurlyeq,F)}$ together with the *multicut rule*, i.e., the following rule, where $n, m \geq 1$ and A^k is an abbreviation for the multiset $A, \ldots, A$ containing exactly k copies of the formula A

$$\frac{\Gamma \Rightarrow \Delta, A^n \quad A^m, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{Mcut}$$

We call the formula A in this application of the multicut rule the *cut formula*. Deriving all the axioms for $\mathcal{L}_{(N,\preccurlyeq,F)}$ in the system $\mathsf{G}_{(N,\preccurlyeq,F)}\mathsf{ConW}$ is straightforward: the axioms for the logics $F(i)$ are derived as in the monomodal case, and the interaction axioms for $i \preccurlyeq j$ are obtained by a single application of the rule k_i. As usual, modus ponens is simulated by applications of the cut rule.

In the second step we show in a rather standard way that the multicut rule can be eliminated from derivations in the system $\mathsf{G}_{(N,\preccurlyeq,F)}\mathsf{ConW}$ (this also follows directly from checking that the system $\mathsf{G}_{(N,\preccurlyeq,F)}\mathsf{ConW}$ satisfies the general criteria for cut elimination in [36, 39]). As usual, the proof is by double induction on the *complexity* of the cut formula, i.e., the number of symbols in the cut formula, and the sum of the depths of the derivations of the two premisses of the application of the multicut rule. Applications of the multicut rule are then pushed upwards into the derivations of the premisses of that application, until in both of the latter at least one occurrence of the cut formula is introduced by the last applied rule, at which point the complexity of the cut formula is reduced. Since the reasoning for the different cases is rather standard, here we only consider two exemplary cases. See, e.g., [67, Sec. 4.1.9] for the reasoning in the propositional cases.

As a first example, consider the multicut below, with applications of rules based on a description such that $i \preccurlyeq j, k$ and $\ell \preccurlyeq i, m, n$ with $\mathsf{KD} \subseteq F(\ell)$, $\mathsf{K4} \subseteq F(j), F(m)$, but with K4 not contained in the other logics.

$$\frac{\dfrac{\Box_j\Gamma_j, \Sigma_j, \Sigma_k, \Sigma_i \Rightarrow A}{\Omega, \Box_j\Gamma_j, \Box_j\Sigma_j, \Box_k\Sigma_k, \Box_i\Sigma_i \Rightarrow \Box_i A, \Xi, \Box_i A^{n-1}}\ \mathsf{k}_i \quad \dfrac{\Box_m\Gamma_m, \Sigma_m, A^s, \Sigma_i, \Sigma_n \Rightarrow}{\Upsilon, \Box_i A^{t-s}, \Box_m\Gamma_m, \Box_m\Sigma_m, \Box_i A^s, \Box_i\Sigma_i, \Box_n\Sigma_n \Rightarrow \Pi}\ \mathsf{d}_\ell}{\Omega, \Box_j\Gamma_j, \Box_j\Sigma_j, \Box_k\Sigma_k, \Upsilon, \Box_i\Sigma_i, \Box_m\Gamma_m, \Box_m\Sigma_m, \Box_i\Sigma_i, \Box_n\Sigma_n \Rightarrow \Xi, \Pi}\ \mathsf{Mcut}$$

As usual, the multicut is replaced with a multicut on the formula of lower complexity A as follows.

$$\frac{\dfrac{\Box_j\Gamma_j, \Sigma_j, \Sigma_k, \Sigma_i \Rightarrow A \qquad \Box_m\Gamma_m, \Sigma_m, A^s, \Sigma_i, \Sigma_n \Rightarrow}{\Box_j\Gamma_j, \Sigma_j, \Sigma_k, \Sigma_i, \Box_m\Gamma_m, \Sigma_m, \Sigma_i, \Sigma_n \Rightarrow}\ \mathsf{Mcut}}{\Omega, \Upsilon, \Box_j\Gamma_j, \Box_j\Sigma_j, \Box_k\Sigma_k, \Box_i\Sigma_i, \Box_m\Gamma_m, \Box_m\Sigma_m, \Box_i\Sigma_i, \Box_n\Sigma_n \Rightarrow \Xi, \Pi}\ \mathsf{d}_\ell$$

Crucially, since the relation $\preccurlyeq$ is transitive, we know that $\ell \preccurlyeq j, k$ as well, which renders the application of the rule d_ℓ at the bottom permissible.

As a second example, consider the multicut with cut formula $\Box_i A$ below, based on a description $(N, \preccurlyeq, F)$ with $k \preccurlyeq i \preccurlyeq j$, such that $\mathsf{K4} \subseteq F(i), F(j)$ and $\mathsf{KD} \subseteq F(k)$.

$$\frac{\dfrac{\Box_j\Gamma_j, \Sigma_j \Rightarrow A}{\Omega, \Box_j\Gamma_j, \Box_j\Sigma_j \Rightarrow \Box_i A, \Xi, \Box_i A^{n-1}}\ \mathsf{k}_i \quad \dfrac{\Box_i A^{\ell-s}, \Box_i\Gamma_i, A^s, \Sigma_i, \Sigma_k \Rightarrow}{\Upsilon, \Box_i A^{\ell-s}, \Box_i\Gamma_i, \Box_i A^s, \Box_i\Sigma_i, \Box_k\Sigma_k \Rightarrow \Pi}\ \mathsf{d}_k}{\Omega, \Box_j\Gamma_j, \Box_j\Sigma_j, \Upsilon, \Box_i\Gamma_i, \Box_i\Sigma_i, \Box_k\Sigma_k \Rightarrow \Xi, \Pi}\ \mathsf{Mcut}$$

$$\frac{\mathcal{S}\left\{\Gamma \Rightarrow \Delta /\!\!/^{j} \Sigma, A \Rightarrow \Pi\right\}}{\mathcal{S}\left\{\Gamma, \Box_i A \Rightarrow \Delta /\!\!/^{j} \Sigma \Rightarrow \Pi\right\}} \ \Box_{ij_L} \qquad \frac{\mathcal{G} /\!\!/^{k} \Gamma \Rightarrow \Delta /\!\!/^{i} \ \Rightarrow A}{\mathcal{G} /\!\!/^{k} \Gamma \Rightarrow \Delta, \Box_i A} \ \Box_{iR}$$

$$\frac{\mathcal{G} /\!\!/^{k} \Gamma \Rightarrow \Delta /\!\!/^{j} A \Rightarrow}{\mathcal{G} /\!\!/^{k} \Gamma, \Box_i A \Rightarrow \Delta} \ \mathsf{d}_{ij} \qquad \frac{\mathcal{S}\{\Gamma, A \Rightarrow \Delta\}}{\mathcal{S}\{\Gamma, \Box_i A \Rightarrow \Delta\}} \ \mathsf{t}_i \qquad \frac{\mathcal{S}\left\{\Gamma \Rightarrow \Delta /\!\!/^{j} \Sigma, \Box_i A \Rightarrow \Pi\right\}}{\mathcal{S}\left\{\Gamma, \Box_i A \Rightarrow \Delta /\!\!/^{j} \Sigma \Rightarrow \Pi\right\}} \ 4_{ij}$$

$$\mathcal{R}_{(N, \preccurlyeq, F)} := \{\Box_{iR} : i \in N\} \cup \{\Box_{ij_L} : i, j \in N, i \in \uparrow(j)\} \cup \{\mathsf{d}_{ij} : i, j \in N, i \in \uparrow^{\mathsf{D}}(j)\}$$
$$\cup \{\mathsf{t}_i : i \in N, \mathsf{KT} \subseteq F(i)\} \cup \{4_{ij} : i, j \in N, i \in \uparrow^{4}(j)\}$$

Fig. 6. The linear nested sequent rules for the simply dependent multimodal logic given by the description $(N, \preccurlyeq, F)$.

This multicut is replaced by two multicuts, an application of d_k and contractions as follows:

$$\dfrac{\dfrac{\Box_j\Gamma_j, \Sigma_j \Rightarrow A \qquad \dfrac{\dfrac{\Box_j\Gamma_j, \Sigma_j \Rightarrow A}{\Box_j\Gamma_j, \Box_j\Sigma_j \Rightarrow \Box_i A}\ \mathsf{k}_i \qquad \Box_i A^{\ell-s}, \Box_i\Gamma_i, A^s, \Sigma_i, \Sigma_k \Rightarrow}{\Box_j\Gamma_j, \Box_j\Sigma_j, \Box_i\Gamma_i, A^s, \Sigma_i, \Sigma_k \Rightarrow}\ \mathsf{Mcut}}{\dfrac{\Box_j\Gamma_j, \Sigma_j, \Box_j\Gamma_j, \Box_j\Sigma_j, \Box_i\Gamma_i, \Sigma_i, \Sigma_k \Rightarrow}{\Omega, \Upsilon, \Box_j\Gamma_j, \Box_j\Sigma_j, \Box_j\Gamma_j, \Box_j\Sigma_j, \Box_i\Gamma_i, \Box_i\Sigma_i, \Box_k\Sigma_k \Rightarrow \Xi, \Pi}\ \mathsf{d}_k}\ \mathsf{Mcut}}{\Omega, \Upsilon, \Box_j\Gamma_j, \Box_j\Sigma_j, \Box_i\Gamma_i, \Box_i\Sigma_i, \Box_k\Sigma_k \Rightarrow \Xi, \Pi}\ \mathsf{Con}$$

The upper multicut is then eliminated using the (inner) induction hypothesis on the sum of the depths of the derivations of its premisses, the lower one is eliminated using the (outer) induction hypothesis on the complexity of the cut formula. Note that for this transformation to work it is crucial that the logic $F(j)$ is also transitive, i.e., that the description $(N, \preccurlyeq, F)$ is transitive closed, since otherwise we would not be able to apply the rule d_k with boxed context formulae $\Box_j\Gamma_j$ and $\Box_j\Sigma_j$. A similar situation occurs if the application of d_k is replaced with an application of k_k with an additional principal formula on the right.

The general cases of the above examples as well as the remaining cases are treated similarly. □

In order to convert the resulting sequent systems into LNS systems, we need to modify the linear nested setting to account for all the different non-invertible right rules. For this, given a description $(N, \preccurlyeq, F)$ we introduce nesting operators $/\!\!/^{i}$ for every $i \in N$, and change the interpretation so that they are interpreted by the corresponding modality:

$$\iota(\Gamma \Rightarrow \Delta) := \bigwedge \Gamma \to \bigvee \Delta$$
$$\iota(\Gamma \Rightarrow \Delta /\!\!/^{i} \mathcal{H}) := \bigwedge \Gamma \to \bigvee \Delta \vee \Box_i\, \iota(\mathcal{H})$$

The modal sequent rules of $\mathsf{G}_{(N, \preccurlyeq, F)}$ are then decomposed into the modal linear nested sequent rules shown in Fig. 6. The propositional rules are those of $\mathsf{LNS_G}$ (Fig. 1). We call the resulting calculus $\mathsf{LNS}_{(N, \preccurlyeq, F)}$. The intuition behind the rules is that an application of the sequent rule k_i is decomposed into an application of the rule $\Box_{iR}$ followed by applications of $\Box_{ij_L}$ to unpack the principal formulae of the sequent rule, and applications of the rule 4_{ij} to move the boxed context formulae into the next component.

Example 3.10. The linear nested sequent calculus for the logic $\mathsf{KT} \oplus_{\subseteq} \mathsf{S4}$ contains the LNS rules $\Box_{11L}, \Box_{21L}, \Box_{22L}, \Box_{1R}, \Box_{2R}, \mathsf{d}_{11}, \mathsf{d}_{21}, \mathsf{d}_{22}, \mathsf{t}_1, \mathsf{t}_2, 4_{21}$, and 4_{22}.

Remark 3. *The previous example illustrates the added modularity of the linear nested sequent approach: if, as in Rem. 2 we wanted to obtain a linear nested sequent calculus for the logic* $\mathsf{KT} \oplus_{\subseteq} \mathsf{KT}$ *from the calculus for* $\mathsf{KT} \oplus_{\subseteq} \mathsf{S4}$ *above, we would only need to delete the rules* 4_{21} *and* 4_{22} *from the rule set, keeping all other rules the same. This is in stark contrast to the modification of almost all modal rules required in the ordinary sequent setting. Note however, that we have modularity, and indeed completeness, only for transitive-closed descriptions. I.e., we would not be able to obtain a calculus for the logic* $\mathsf{S4} \oplus_{\subseteq} \mathsf{KT}$*, since it is not given by a transitive-closed description.*

Theorem 3.11. *If* $(N, \preccurlyeq, F)$ *is a transitive-closed description of a simply dependent multimodal logic, then* $\mathsf{LNS}_{(N,\preccurlyeq,F)}\mathsf{ConW}$ *is sound and complete for the logic given by* $(N, \preccurlyeq, F)$*, i.e., for every formula* A *we have that* $A \in \mathcal{L}_{(N,\preccurlyeq,F)}$ *if and only if* $\vdash_{\mathsf{LNS}_{(N,\preccurlyeq,F)}\mathsf{ConW}} \Rightarrow A$.

Proof. For soundness, i.e., the "if" statement, we show that whenever the negation of the interpretation of the conclusion of a rule from $\mathsf{LNS}_{(N,\preccurlyeq,F)}\mathsf{ConW}$ is satisfiable in a $(N, \preccurlyeq, F)$-frame, then so is the negation of the interpretation of at least one of its premisses. This makes essential use of the fact that in such frames we have $R_i \subseteq R_j$ whenever $i \preccurlyeq j$. For completeness, i.e., the "only if" statement, we again simulate the sequent rules in the last components, i.e., we translate a sequent derivation in $\mathsf{G}_{(N,\preccurlyeq,F)}\mathsf{ConW}$ bottom-up into a linear nested sequent derivation in $\mathsf{LNS}_{(N,\preccurlyeq,F)}$, simulating propositional sequent rules by their linear nested sequent counterparts, and modal sequent rules by a number of applications of the corresponding linear nested sequent rules. E.g., an application of the modal sequent rule d_i with history (i.e., trace to the conclusion of the sequent derivation) captured by the linear nested sequent $\mathcal{G}$ is simulated as follows (assuming that $k \in \uparrow^{\neg 4}(i)$):

$$\dfrac{\{\Box_j\Gamma_j, \Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^{\neg 4}(i)\}, A \Rightarrow}{\Omega, \{\Box_j\Gamma_j, \Box_j\Sigma_j : j \in \uparrow^4(i)\}, \{\Box_j\Sigma_j : j \in \uparrow^{\neg 4}(i)\}, \Box_k A \Rightarrow \Xi}\ \mathsf{d}_i$$
$$\vdots\ \mathcal{G}$$

$$\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\dfrac{\mathcal{G} \mathbin{/\!\!/} \Omega, \Rightarrow \Xi \mathbin{/\!\!/}^i \{\Box_j\Gamma_j, \Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^{\neg 4}(i)\}, A \Rightarrow}{\mathcal{G} \mathbin{/\!\!/} \Omega, \{\Box_j\Gamma_j : j \in \uparrow^4(i)\}, \Rightarrow \Xi \mathbin{/\!\!/}^i \{\Sigma_j : j \in \uparrow^4(i)\}, \{\Sigma_j : j \in \uparrow^{\neg 4}(i)\}, A \Rightarrow}\ 4_{ji}}{\mathcal{G} \mathbin{/\!\!/} \Omega, \{\Box_j\Gamma_j, \Box_j\Sigma_j : j \in \uparrow^4(i)\} \Rightarrow \Xi \mathbin{/\!\!/}^i \{\Sigma_j : j \in \uparrow^{\neg 4}(i)\}, A \Rightarrow}\ \Box_{ji_L}}{\mathcal{G} \mathbin{/\!\!/} \Omega, \{\Box_j\Gamma_j, \Box_j\Sigma_j : j \in \uparrow^4(i)\}, \{\Box_j\Sigma_j : j \in \uparrow^{\neg 4}(i)\} \Rightarrow \Xi \mathbin{/\!\!/}^i A \Rightarrow}\ \Box_{ji_L}}{\mathcal{G} \mathbin{/\!\!/} \Omega, \{\Box_j\Gamma_j, \Box_j\Sigma_j : j \in \uparrow^4(i)\}, \{\Box_j\Sigma_j : j \in \uparrow^{\neg 4}(i)\}, \Box_k A \Rightarrow \Xi}\ \mathsf{d}_{ki}$$

The remaining modal rules are simulated in a similar way. □

Note that the proof of completeness via simulation of the sequent calculus in the last component actually shows a slightly stronger statement, i.e., completeness for a variant of the calculus where the rules are restricted so they only manipulate the last components. More precisely:

Definition 3.12. An application of a linear nested sequent rule is *end-active* if the rightmost components of the premisses are active and the only active components (in premiss and conclusion) are the two rightmost ones. The *end-active variant* of a LNS calculus is the calculus with the rules restricted to end-active applications.

Example 3.13. The application of the rule $\wedge_L$ below left is end-active, the one below right is not, since the rightmost component is not active.

$$\dfrac{\mathcal{G} \mathbin{/\!\!/} \Gamma, A, B \Rightarrow \Delta}{\mathcal{G} \mathbin{/\!\!/} \Gamma, A \wedge B \Rightarrow \Delta}\ \wedge_L \qquad\qquad \dfrac{\mathcal{G} \mathbin{/\!\!/} \Gamma, A, B \Rightarrow \Delta \mathbin{/\!\!/} \Sigma \Rightarrow \Pi}{\mathcal{G} \mathbin{/\!\!/} \Gamma, A \wedge B \Rightarrow \Delta \mathbin{/\!\!/} \Sigma \Rightarrow \Pi}\ \wedge_L$$

Applications of the modal rules in the LNS calculi for non-normal modal logics considered in this paper (see next section) are always end-active. An application of the modal rule

$$\frac{\mathcal{S}\{\Gamma \Rightarrow \Delta \,/\!\!/\, \Sigma, A \Rightarrow \Pi\}}{\mathcal{S}\{\Gamma, \Box A \Rightarrow \Delta \,/\!\!/\, \Sigma \Rightarrow \Pi\}}\ \Box_L$$

is end-active only if $\Sigma \Rightarrow \Pi$ is the rightmost component.

Corollary 3.14. *If $(N, \preccurlyeq, F)$ is a transitive-closed description of a simply dependent multimodal logic, then the end-active variant of $\mathsf{LNS}_{(N,\preccurlyeq,F)}\mathsf{ConW}$ is sound and complete for the logic given by $(N, \preccurlyeq, F)$, i.e., for every formula A we have $A \in \mathcal{L}_{(N,\preccurlyeq,F)}$ if and only if $\Rightarrow A$ is derivable in the end-active variant of $\mathsf{LNS}_{(N,\preccurlyeq,F)}\mathsf{ConW}$.*

Proof. Soundness, i.e., the "if" statement, follows immediately from soundness for the full calculus. For completeness, i.e., the "only if" statement, observe that the sequent rules are simulated in the last component, i.e., by end-active applications of the linear nested sequent rules. □

The fact that we can restrict the linear nested calculi to their end-active variants will be exploited in Section 5 for reducing the search space in proof search.

The example of simply dependent multimodal logics shows another conceptual advantage of LNS calculi over standard sequent calculi: for more involved sequent calculi such as the ones in Fig. 5 the decomposition of the sequent rules into their different components tends to make the corresponding LNS calculi (Fig. 6) a lot more readable. Of course the previous theorem also shows that the obvious adaption of this calculus to the full nested sequent setting of [6, 62] is sound and cut-free complete for the corresponding logic.

3.2 Non-normal modal logics

The same ideas also yield LNS calculi for some *non-normal* modal logics, i.e., modal logics that are not extensions of modal logic K (see [10] for an introduction). The calculi themselves are of independent interest since, to the best of our knowledge, nested sequent calculi for the logics below have not been considered before in the literature. The most basic non-normal logic, *classical modal logic* E, is given Hilbert-style by extending the axioms and rules for classical propositional logic with only the *congruence rule* (E) for the $\Box$ connective

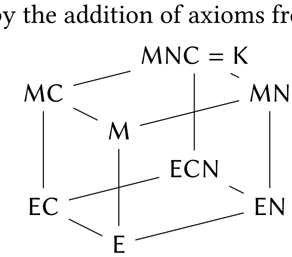
$$\frac{A \to B \quad B \to A}{\Box A \to \Box B}\ (\mathsf{E})$$

which allows exchanging logically equivalent formulae under the modality. Some of the better known extensions of this logic are formulated by the addition of axioms from the list below left.

M $\quad \Box(A \wedge B) \to (\Box A \wedge \Box B)$

C $\quad (\Box A \wedge \Box B) \to \Box(A \wedge B)$

N $\quad \Box\top$



Together, these extensions form what could be termed the *classical cube* (above right). Note that the extension of E with all the axioms M, C, N is normal modal logic K. Fig. 7 shows the modal rules of the standard cut-free sequent calculi for these logics, where in addition weakening is embedded in the conclusion. Extensions of E are written by concatenating the names of the axioms, and in presence of the monotonicity axiom M, sometimes the initial E is dropped. E.g., the logic EMC = MC is the extension of E with axioms M and C. Its sequent calculus G_{MC} is given by the

$$\frac{A \Rightarrow B \quad B \Rightarrow A}{\Gamma, \Box A \Rightarrow \Box B, \Delta}\ (\mathsf{E}) \qquad \frac{A \Rightarrow B}{\Gamma, \Box A \Rightarrow \Box B, \Delta}\ (\mathsf{M}) \qquad \frac{\Rightarrow A}{\Gamma \Rightarrow \Box A, \Delta}\ (\mathsf{N})$$

$$\frac{A_1, \ldots, A_n \Rightarrow B \quad B \Rightarrow A_1 \quad \cdots \quad B \Rightarrow A_n}{\Gamma, \Box A_1, \ldots, \Box A_n \Rightarrow \Box B, \Delta}\ (\mathsf{E}n) \qquad \frac{A_1, \ldots, A_n \Rightarrow B}{\Gamma, \Box A_1, \ldots, \Box A_n \Rightarrow \Box B, \Delta}\ (\mathsf{M}n)$$

$\mathsf{G_E}$	$\{(\mathsf{E})\}$	$\mathsf{G_{EN}}$	$\{(\mathsf{E}), (\mathsf{N})\}$	$\mathsf{G_{EC}}$	$\{(\mathsf{E}n) : n \geq 1\}$	$\mathsf{G_{ECN}}$	$\{(\mathsf{E}n) : n \geq 1\} \cup \{(\mathsf{N})\}$
$\mathsf{G_M}$	$\{(\mathsf{M})\}$	$\mathsf{G_{MN}}$	$\{(\mathsf{M}), (\mathsf{N})\}$	$\mathsf{G_{MC}}$	$\{(\mathsf{M}n) : n \geq 1\}$	$\mathsf{G_{MCN}}$	$\{(\mathsf{M}n) : n \geq 0\}$

Fig. 7. Sequent rules and calculi for some non-normal modal logics

$$\frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/_{\mathsf{e}} (\Rightarrow B; B \Rightarrow)}{\mathcal{G} /\!/ \Gamma \Rightarrow \Box B, \Delta}\ \Box_R^{\mathsf{e}} \qquad \frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ \Sigma, A \Rightarrow \Pi \quad \mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ \Omega \Rightarrow A, \Theta}{\mathcal{G} /\!/ \Gamma, \Box A \Rightarrow \Delta /\!/_{\mathsf{e}} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)}\ \Box_L^{\mathsf{e}}$$

$$\frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ \Rightarrow B}{\mathcal{G} /\!/ \Gamma \Rightarrow \Box B, \Delta}\ \mathsf{N} \qquad \frac{\mathcal{G} /\!/_{\mathsf{e}} (\Sigma \Rightarrow \Pi; \Omega, \bot \Rightarrow \Theta)}{\mathcal{G} /\!/_{\mathsf{e}} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)}\ \mathsf{M}$$

$$\frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/_{\mathsf{e}} (\Sigma, A \Rightarrow \Pi; \Omega \Rightarrow \Theta) \quad \mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ \Omega \Rightarrow A, \Theta}{\mathcal{G} /\!/ \Gamma, \Box A \Rightarrow \Delta /\!/_{\mathsf{e}} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)}\ \mathsf{C}$$

$$\mathsf{LNS}_{\mathsf{E}\mathcal{A}} : \quad \{\Box_R, \Box_L\} \cup \mathcal{A} \quad \text{for } \mathcal{A} \subseteq \{\mathsf{N}, \mathsf{M}, \mathsf{C}\}$$

Fig. 8. Linear nested sequent calculi for non-normal modal logics

standard propositional rules of G_r (see Def. 3.6) together with the rule (E) and the rules $(\mathsf{M}n)$ for $n \geq 1$. For all of the logics apart from EN and ECN these calculi were given in [35], the one for EN is from [28], the remaining calculus is an easy extension. It is not too difficult to show admissibility of weakening and contraction in these calculi. But since the calculi originally were considered for sequents based on sets instead of multisets, and in preparation for later results, we mostly consider their extensions with the structural rules. E.g., we consider $\mathsf{G_{EM}ConW}$ instead of $\mathsf{G_{EM}}$.

In order to construct linear nested calculi for these logics, again we would like to decompose the sequent rules from Fig. 7 into their different components. However, there are two complications compared to the case of normal modal logics: we need a mechanism for capturing the fact that e.g. in the rule (M) exactly one boxed formula is introduced on the left hand side; and we need a way of handling multiple premisses of rules such as (E) and $(\mathsf{E}n)$. We solve the first problem by introducing an auxiliary nesting operator $/\!/_{\mathsf{e}}$ to capture a state where a sequent rule has been *partly processed*, i.e., where the simulation of the sequent rule is still unfinished. The intuition is that in this "partly processed" state, only other LNS rules continuing or eventually finishing the simulation of the original sequent rule can be applied. In contrast, the intuition for the original nesting $/\!/$ is that the simulation of a rule is finished. We restrict the occurrence of $/\!/_{\mathsf{e}}$ to the last components.

To solve the problem of multiple premisses, we make the nesting operator $/\!/_{\mathsf{e}}$ *binary*, which permits the storage of more information about the premisses. In particular, we can now store the two "directions" of implications given, e.g., in the premisses of rule (E). Linear nested sequents for classical non normal modal logics are then given by:

$$\mathsf{LNS_e} ::= \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta /\!/_{\mathsf{e}} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta) \mid \Gamma \Rightarrow \Delta /\!/ \mathsf{LNS_e}$$

Fig. 8 shows the modal rules for these logics. For a logic $\mathsf{E}\mathcal{A}$ with $\mathcal{A} \subseteq \{\mathsf{N}, \mathsf{M}, \mathsf{C}\}$ the calculus $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}$ then contains the corresponding modal rules along with the propositional rules of $\mathsf{LNS_G}$ (Fig. 1) with the restriction that they are not applied inside the nesting $/\!/_{\mathsf{e}}$. To keep the presentation simple, we slightly abuse notation and write e.g. M both for the axiom and the corresponding rule.

Remark 4. *At first sight the linear nested sequent rules of Fig. 8 might look more complicated than the ordinary sequent rules of Fig. 7. Moreover, it could be argued that the latter systems could be considered to be modular, since, e.g., to obtain a calculus for the logic* MC *from the logic* EC *it would be sufficient to simply* add *the rules* $\{(\mathsf{M}_n) : n \geq 1\}$ *to the system for* EC*, so it might not be clear immediately what we have gained by moving to the nested sequent setting. But it is worth noting that, for every extension of* EC*, the sequent systems of Fig. 7 consist of an infinite number of rules, where in particular every rule has a different number of principal formulae. In contrast, the linear nested sequent systems of Fig. 8 each consist of a finite number of rules, where each rule has at most one principal formula. Moreover, each of the axioms* N, M, C *corresponds to exactly one additional rule in the linear nested sequent setting. Both of these properties are philosophically highly relevant, e.g., from the point of view of* proof-theoretic semantics. *For more details on this, see, e.g., [69, Sec. 1.2 and 1.3].*

Theorem 3.15 (Completeness). *The linear nested sequent calculi of Fig. 8 are complete w.r.t. the corresponding logics, i.e., if $A \in \mathsf{E}\mathcal{A}$, then $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{ConW}} \Rightarrow A$.*

Proof. Again the proof is via simulation of the sequent calculi. An application of the rule ($\mathsf{E}n$) is simulated by the following derivation:

$$\dfrac{\dfrac{\dfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A_1, \dots, A_n \Rightarrow B \qquad \mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ B \Rightarrow A_n}{\mathcal{G} /\!\!/ \Gamma, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (A_1, \dots, A_{n-1} \Rightarrow B; B \Rightarrow)}\ \Box_L^{\mathsf{e}} \\ \vdots \\ \mathcal{G} /\!\!/ \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (A_1 \Rightarrow B; B \Rightarrow) \qquad \mathcal{G} /\!\!/ \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta /\!\!/ B \Rightarrow A_1}{\mathcal{G} /\!\!/ \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (\Rightarrow B; B \Rightarrow)}\ \mathsf{C}}{\mathcal{G} /\!\!/ \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta}\ \Box_R^{\mathsf{e}}$$

Here the vertical dots abbreviate successive applications of the rule C with right premisses $\mathcal{G} /\!\!/ \Gamma, \Box A_{i+1}, \dots, \Box A_n \Rightarrow \Delta /\!\!/ B \Rightarrow A_i$ for $1 < i < n$. The case of $n = 1$ gives the simulation of the rule (E). The sequent rule N is simulated directly by the LNS rule N. In the monotone case the simulations are essentially the same, but after creating the new nesting using the $\Box_R^{\mathsf{e}}$ rule (bottom-up) we first apply the rule M to add $\bot$ and thus make all the premisses for the "backwards direction", i.e., the implications $B \Rightarrow A_i$, trivially derivable. The sequent rule ($\mathsf{M}n$) then is simulated by

$$\dfrac{\dfrac{\dfrac{\dfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A_1, \dots, A_n \Rightarrow B \qquad \overline{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ B, \bot \Rightarrow A_n}\ \bot_L}{\mathcal{G} /\!\!/ \Gamma, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (A_1, \dots, A_{n-1} \Rightarrow B; B, \bot \Rightarrow)}\ \Box_L^{\mathsf{e}} \\ \vdots \\ \mathcal{G} /\!\!/ \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (A_1 \Rightarrow B; B, \bot \Rightarrow) \qquad \overline{\mathcal{G} /\!\!/ \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta /\!\!/ B, \bot \Rightarrow}\ \bot_L}{\mathcal{G} /\!\!/ \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (\Rightarrow B; B, \bot \Rightarrow)}\ \mathsf{C}}{\mathcal{G} /\!\!/ \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (\Rightarrow B; B \Rightarrow)}\ \mathsf{M}}{\mathcal{G} /\!\!/ \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta}\ \Box_R^{\mathsf{e}}$$

Again, the vertical dots abbreviate applications of the rule C, and the case of $n = 1$ gives the simulation of the rule (M). □

As for simply dependent multimodal logics, the completeness proof via simulation of the sequent rules in the last component also shows completeness of the end-active variants of the calculi.

Corollary 3.16. *The end-active variants of the linear nested sequent calculi of Fig. 8 are complete w.r.t. the corresponding logics, i.e., if $A \in \mathsf{E}\mathcal{A}$, then $\Rightarrow A$ is derivable in the end-active variant of* $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{ConW}$.

For showing soundness of such calculi we need a different method, though. This is due to the fact that, unlike for normal modal logics, there is no clear formula interpretation for linear nested sequents for non-normal modal logics. However, since the propositional rules cannot be applied inside the auxiliary nesting $/\!\!/_{\mathsf{e}}$, the modal rules only occur in blocks which can be seen as a macro-rule corresponding to a modal sequent rule. In addition, we will show by a permutation-of-rules argument that we can restrict the propositional rules to end-active applications (see Def. 3.12). Soundness of the full calculus then follows from soundness of the end-active variant, which is shown by translating derivations back into derivations in the corresponding sequent calculus.

Similar to the argument for *levelled derivations* in [43, p. 241, Prop. 2], the following Lemmata show that the propositional rules can be restricted to be end-active. The first step is to show invertibility of the general forms of the propositional rules. Since all our calculi include the contraction rule we show this in a slightly more general form.

Definition 3.17. If $\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \dots /\!\!/ \Gamma_n \Rightarrow \Delta_n$ is a linear nested sequent, then the *level* of the occurrences of formulae in Γ_i, Δ_i is i.

In all the results stated in this section, we will assume that $\mathcal{A} \subseteq \{\mathsf{N}, \mathsf{M}, \mathsf{C}\}$.

Lemma 3.18 (Admissibility of Weakening). *The weakening rules* $\mathsf{W}_L, \mathsf{W}_R$ *are depth-preserving admissible in* $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}$ *and* $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}$, *i.e., if there is a derivation* $\mathcal{D}$ *of* $\mathcal{S}\{\Gamma \Rightarrow \Delta\}$ *with depth at most* n, *then there are derivations* $\mathcal{D}_1$ *and* $\mathcal{D}_2$ *of* $\mathcal{S}\{\Gamma, A \Rightarrow \Delta\}$ *and* $\mathcal{S}\{\Gamma \Rightarrow A, \Delta\}$ *respectively with depth at most* n *in the same system. Moreover, if the level of the active components of every rule application in* $\mathcal{D}$ *is at least* k, *then the same holds for* $\mathcal{D}_1$ *and* $\mathcal{D}_2$.

Proof. As usual by induction on the depth of the derivation: applications of weakening are permuted upwards over every rule until they are absorbed by the initial sequents. Since this does not change the structure of the derivation and in particular does not introduce any new rule applications, the depth of the derivation and the minimal level of the active components of the rule applications is preserved. □

Lemma 3.19 (Multi-invertibility of the propositional rules). *The non-end-active versions of the propositional rules are* m-invertible *in* $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}$, *i.e., for every* $n \geq 1$ *we have:*

(1) If $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\{\Gamma, (\neg A)^n \Rightarrow \Delta\}$, *then* $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\left\{\Gamma \Rightarrow A^{n+k}, \Delta\right\}$ *for some* $k \geq 0$.

(2) If $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\{\Gamma \Rightarrow (\neg A)^n, \Delta\}$, *then* $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\left\{\Gamma, A^{n+k} \Rightarrow \Delta\right\}$ *for some* $k \geq 0$.

(3) If $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\{\Gamma, (A \to B)^n \Rightarrow \Delta\}$, *then* $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\left\{\Gamma, B^{n+k} \Rightarrow \Delta\right\}$ *and* $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\left\{\Gamma \Rightarrow A^{n+\ell}, \Delta\right\}$ *for some* $k, \ell \geq 0$.

(4) If $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\{\Gamma \Rightarrow (A \to B)^n, \Delta\}$, *then* $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\left\{\Gamma, A^{n+k} \Rightarrow B^{n+\ell}, \Delta\right\}$ *for some* $k, \ell \geq 0$.

(5) If $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\{\Gamma, (A \lor B)^n \Rightarrow \Delta\}$, *then* $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\left\{\Gamma, A^{n+k} \Rightarrow \Delta\right\}$ *and* $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\left\{\Gamma, B^{n+\ell} \Rightarrow \Delta\right\}$ *for some* $k, \ell \geq 0$.

(6) If $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\{\Gamma \Rightarrow (A \lor B)^n, \Delta\}$, *then* $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\left\{\Gamma \Rightarrow A^{n+k}, B^{n+\ell}, \Delta\right\}$ *for some* $k, \ell \geq 0$.

(7) If $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\{\Gamma, (A \land B)^n \Rightarrow \Delta\}$, *then* $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\left\{\Gamma, A^{n+k}, B^{n+\ell} \Rightarrow \Delta\right\}$ *for some* $k, \ell \geq 0$.

(8) If $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\{\Gamma \Rightarrow (A \land B)^n, \Delta\}$, *then* $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\left\{\Gamma \Rightarrow A^{n+k}, \Delta\right\}$ *and* $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}} \mathcal{S}\left\{\Gamma \Rightarrow B^{n+\ell}, \Delta\right\}$ *for some* $k, \ell \geq 0$.

Moreover, both the depth of the derivation and the minimal level of the active components of rule applications are preserved.

Proof. By induction on the depth of the derivation, distinguishing cases according to the last applied rule. E.g., for the rule $\to_R$ we have: if $\mathcal{S}\{\Gamma \Rightarrow (A \to B)^n, \Delta\}$ is an initial sequent or the conclusion of one of the rules $\bot_L$ or $\top_R$, then so is the linear nested sequent $\mathcal{S}\left\{\Gamma, A^{n+k} \Rightarrow B^{n+\ell}, \Delta\right\}$ for any $k, \ell \geq 0$. If the last applied rule was not a contraction rule, we apply the induction hypothesis to its premiss(es), followed by the same rule. E.g., if the last applied rule was the rule C, and the component containing $(A \rightarrow B)^n$ is the penultimate one, we have a derivation ending in

$$\dfrac{\begin{matrix}\vdots \\ \mathcal{G} \mathbin{/\!/} \Gamma' \Rightarrow (A \rightarrow B)^n, \Delta \mathbin{/\!/}_{\mathsf{e}} (\Sigma, C \Rightarrow \Pi; \Omega \Rightarrow \Theta)\end{matrix} \qquad \begin{matrix}\vdots \\ \mathcal{G} \mathbin{/\!/} \Gamma' \Rightarrow (A \rightarrow B)^n, \Delta \mathbin{/\!/} \Omega \Rightarrow C, \Theta\end{matrix}}{\mathcal{G} \mathbin{/\!/} \Gamma', \Box C \Rightarrow (A \rightarrow B)^n \Delta \mathbin{/\!/}_{\mathsf{e}} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)}\ \mathsf{C}$$

Using the induction hypothesis, for some i, j, k, ℓ we obtain derivations of $\mathcal{G} \mathbin{/\!/} \Gamma', A^{n+i} \Rightarrow B^{n+j}, \Delta \mathbin{/\!/}_{\mathsf{e}} (\Sigma, C \Rightarrow \Pi; \Omega \Rightarrow \Theta)$ and $\mathcal{G} \mathbin{/\!/} \Gamma', A^{n+k} \Rightarrow B^{n+\ell}, \Delta \mathbin{/\!/} \Omega \Rightarrow C, \Theta$ and admissibility of weakening (Lemma 3.18) followed by an application of C yields the desired $\mathcal{G} \mathbin{/\!/} \Gamma', \Box C, A^{n+\max\{i,k\}} \Rightarrow B^{n+\max\{j,\ell\}}, \Delta \mathbin{/\!/}_{\mathsf{e}} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)$. Finally, if the last applied rule was the contraction rule, we simply apply the induction hypothesis to its premiss. E.g., if the contracted formula is $A \rightarrow B$ and we have a derivation ending in

$$\dfrac{\mathcal{S}\left\{\Gamma \Rightarrow (A \rightarrow B)^{n+1}, \Delta\right\}}{\mathcal{S}\{\Gamma \Rightarrow (A \rightarrow B)^n, \Delta\}}\ \mathsf{C}_R$$

we use the induction hypothesis to obtain $\mathcal{S}\left\{\Gamma, A^{n+1+k} \Rightarrow B^{n+1+\ell}, \Delta\right\}$ for some $k, \ell \geq 0$. □

Of course, setting $n = 1$ in the statement of the previous lemma and (possibly) applying a number of contractions to the result recovers standard invertibility of the propositional rules, albeit not the depth-preserving version.

Using this we first obtain soundness of the full calculus with contraction with respect to the end-active variant.

Lemma 3.20. *If a linear nested sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}$, then it is derivable in the end-active variant of $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}$.*

Proof. Due to the nature of the modal rules it is clear that in a derivation only applications of the propositional rules and contraction can violate the end-activeness condition. We then successively transform a derivation of $\Gamma \Rightarrow \Delta$ into an end-active derivation as follows. Take the bottom-most block of modal rules such that there is an application of a propositional rule or contraction above it with level of the active component smaller than the maximal level of the active components in the modal block. Since the modal rules only apply to formulae in the last component, all such applications of propositional rules introduce a propositional connective which in the conclusion of the modal block is not under a modality. Using multi-invertibility of the propositional connectives (Lemma 3.19) we replace every such formula in the conclusion of the modal block by its constituents, possibly with multiplicity more than one. E.g, if the conclusion of the modal block has the form

$$\dfrac{\begin{matrix}\vdots \\ \mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow A \rightarrow B, \Delta \mathbin{/\!/} \mathcal{H} \mathbin{/\!/} \Sigma \Rightarrow \Pi \mathbin{/\!/}_{\mathsf{e}} (\ \Rightarrow C; C \Rightarrow\ )\end{matrix}}{\mathcal{G} \mathbin{/\!/} \Gamma \Rightarrow A \rightarrow B, \Delta \mathbin{/\!/} \mathcal{H} \mathbin{/\!/} \Sigma \Rightarrow \Box C, \Pi}\ \Box_R^{\mathsf{e}}$$

with the formula $A \rightarrow B$ introduced above the modal block, using m-invertibility we obtain

$$\dfrac{\begin{matrix}\vdots\ \mathcal{D} \\ \mathcal{G} \mathbin{/\!/} \Gamma, A^{1+k} \Rightarrow B^{1+\ell}, \Delta \mathbin{/\!/} \mathcal{H} \mathbin{/\!/} \Sigma \Rightarrow \Pi \mathbin{/\!/}_{\mathsf{e}} (\ \Rightarrow C; C \Rightarrow\ )\end{matrix}}{\mathcal{G} \mathbin{/\!/} \Gamma, A^{1+k} \Rightarrow B^{1+\ell}, \Delta \mathbin{/\!/} \mathcal{H} \mathbin{/\!/} \Sigma \Rightarrow \Box C, \Pi}$$

Then we delete every application of the contraction rule with active component of level smaller than the maximal level of the active components in the modal block from the derivation, possibly using Lemma 3.18 to ensure that the contexts in two-premiss rules are the same. From the proof of Lemma 3.19 it can be seen that afterwards the minimal level of the active components in rule applications in the derivation up to the conclusion of the modal block is at least the maximal level of the active components in the modal block itself. Finally, we use end-active applications of contraction to remove unwanted duplicates followed by end-active applications of the propositional rules to reintroduce the propositional connectives in the right place, i.e., when the component containing the constituent formulae is the last one. Since the conclusion of the original derivation contained only a single component, this is always possible. ◻

From this we obtain soundness of the full calculus by first translating derivations into derivations in the end-active variant, then into derivations in the corresponding sequent calculus:

Theorem 3.21 (Soundness). *If a sequent* $\Gamma \Rightarrow \Delta$ *is derivable in* $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}$ *for* $\mathcal{A} \subseteq \{\mathsf{N}, \mathsf{M}, \mathsf{C}\}$, *then it is derivable in the corresponding sequent calculus* $\mathsf{G}_{\mathsf{E}\mathcal{A}}\mathsf{Con}$. *Hence if* $\vdash_{\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{ConW}} \Rightarrow A$, *then* $A \in \mathsf{E}\mathcal{A}$.

Proof. From the previous lemma we obtain that if a sequent $\Gamma \Rightarrow \Delta$ is derivable in $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}$, then it is derivable in the end-active variant of $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{Con}$. A derivation of the latter form then is translated into a $\mathsf{G}_{\mathsf{E}\mathcal{A}}\mathsf{Con}$ derivation, discarding everything apart from the last component of the linear nested sequents, and translating blocks of modal rules into the corresponding modal sequent rules. E.g., a block consisting of an application of $\Box^{\mathsf{e}}_L$ followed by n applications of C and an application of $\Box^{\mathsf{e}}_R$ is translated into an application of the rule ($\mathsf{E}n$). In the monotone case we use the fact that the rule M permutes down over the rule C, i.e., a modal block

$$\dfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A_1, \dots, A_n \Rightarrow B \qquad \dfrac{}{\mathcal{G} /\!\!/ \Gamma, A_n \Rightarrow \Delta /\!\!/ B, \bot \Rightarrow A_n}\,\bot_L}{\mathcal{G} /\!\!/ \Gamma, A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (A_1, \dots, A_{n-1} \Rightarrow B; B, \bot \Rightarrow)}\,\Box^{\mathsf{e}}_L$$

$$\vdots$$

$$\dfrac{\mathcal{G} /\!\!/ \Gamma, \Box A_{k+1}, \dots, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (A_1, \dots, A_k \Rightarrow B; B, \bot \Rightarrow)}{\mathcal{G} /\!\!/ \Gamma, \Box A_{k+1}, \dots, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (A_1, \dots, A_k \Rightarrow B; B \Rightarrow)}\,\mathsf{M}$$

$$\vdots$$

$$\dfrac{\dfrac{\mathcal{G} /\!\!/ \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (A_1 \Rightarrow B; B \Rightarrow) \qquad \mathcal{G} /\!\!/ \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta /\!\!/ B \Rightarrow A_1}{\mathcal{G} /\!\!/ \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (\Rightarrow B; B \Rightarrow)}\,\mathsf{C}}{\mathcal{G} /\!\!/ \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta}\,\Box^{\mathsf{e}}_R$$

is first turned into the following block by permuting the rule M downwards and closing the derivations of the superfluous premisses using the $\bot_L$ rule:

$$\dfrac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ A_1, \dots, A_n \Rightarrow B \qquad \dfrac{}{\mathcal{G} /\!\!/ \Gamma, A_n \Rightarrow \Delta /\!\!/ B, \bot \Rightarrow A_n}\,\bot_L}{\mathcal{G} /\!\!/ \Gamma, A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (A_1, \dots, A_{n-1} \Rightarrow B; B, \bot \Rightarrow)}\,\Box^{\mathsf{e}}_L$$

$$\vdots$$

$$\dfrac{\dfrac{\dfrac{\mathcal{G} /\!\!/ \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (A_1 \Rightarrow B; B, \bot \Rightarrow) \qquad \dfrac{}{\mathcal{G} /\!\!/ \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta /\!\!/ B, \bot \Rightarrow A_1}\,\bot_L}{\mathcal{G} /\!\!/ \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (\Rightarrow B; B, \bot \Rightarrow)}\,\mathsf{C}}{\mathcal{G} /\!\!/ \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{e}} (\Rightarrow B; B \Rightarrow)}\,\mathsf{M}}{\mathcal{G} /\!\!/ \Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta}\,\Box^{\mathsf{e}}_R$$

The resulting modal block then is translated into an application of the rule $(\mathsf{M}n)$ with premiss $A_1, \dots, A_n \Rightarrow B$ and conclusion $\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta$. The propositional rules only work on the last component, never inside the nesting $/\!\!/_{\mathsf{e}}$ and are translated easily by the corresponding sequent rules.

Soundness of the system $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}\mathsf{ConW}$ then follows from this using admissibility of weakening (Lem. 3.18) and soundness of the sequent system $\mathsf{G}_{\mathsf{E}\mathcal{A}}\mathsf{Con}$ w.r.t. $\mathsf{E}\mathcal{A}$. □

Note that due to the following lemma for the logics of the non-normal cube we could have avoided the complications arising from including the contraction rules in the calculi. However, in view of the calculi in later sections and the fact that the original sequent systems include contraction explicitly or implicitly in the structure of sequents as defined by sets instead of multisets we chose the given more general method for proving soundness.

Lemma 3.22 (Admissibility of Contraction). *Contraction is admissible in the calculus* $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}$*, that is, if there is a derivation* $\mathcal{D}$ *of* $\mathcal{S}\{\Gamma, A, A \Rightarrow \Delta\}$ *(resp.* $\mathcal{S}\{\Gamma \Rightarrow \Delta, A, A\}$*) in* $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}$ *, then there is a derivation* $\mathcal{D}'$ *of* $\mathcal{S}\{\Gamma, A \Rightarrow \Delta\}$ *(resp.* $\mathcal{S}\{\Gamma \Rightarrow \Delta, A\}$*) in* $\mathsf{LNS}_{\mathsf{E}\mathcal{A}}$*.*

Proof. The proof is for both statements simultaneously by double induction on the complexity of the contracted formula and the depth of the derivation. In case the main connective of the contracted formula is a propositional connective, we use invertibility of the propositional rules (Lemma 3.19) followed by the (outer) induction hypothesis on the complexity of the contracted formula. The cases where A is a modal formula and not principal in the last applied rule are dealt with in the standard way by appealing to the (inner) induction hypothesis on the depth of the derivation. If the contracted formula is a modal formula and principal in the last applied rule, we distinguish cases according to the last applied rule. Suppose, e.g., that $\mathcal{S}\{\Gamma, \Box A, \Box A \Rightarrow \Delta\}$ has a derivation of the shape

$$\dfrac{\overset{\pi_1}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/_{\mathsf{e}} (\Sigma, A \Rightarrow \Pi; \Omega \Rightarrow \Theta)} \quad \overset{\pi_2}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/ \Omega \Rightarrow A, \Theta}}{\mathcal{G} /\!\!/ \Gamma, \Box A, \Box A \Rightarrow \Delta /\!\!/_{\mathsf{e}} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)}\ \mathsf{C}$$

Note that the $\Box A$ in the penultimate component of the conclusion of π_2 will be necessarily weakened, since no logical rules can act on it. In the derivation π_1, either $\Box A$ is never active, in which case it can be weakened, or it is active via one of the rules C or $\Box_L^{\mathsf{e}}$. Let us consider the first case:

$$\dfrac{\overset{\pi_1'}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/_{\mathsf{e}} (\Sigma, A, A \Rightarrow \Pi; \Omega \Rightarrow \Theta)} \quad \overset{\pi_1''}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Omega \Rightarrow A, \Theta}}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/_{\mathsf{e}} (\Sigma, A \Rightarrow \Pi; \Omega \Rightarrow \Theta)}\ \mathsf{C}$$

Observe that the modal block in π_1' will eventually end by producing leaves of the form $\mathcal{G} /\!\!/ \Gamma' \Rightarrow \Delta' /\!\!/ \Sigma', A, A \Rightarrow \Pi'$ and $\Omega' \Rightarrow \Theta'$. By induction hypothesis, for every derivation for a sequent of the first form there is a derivation of $\mathcal{G} /\!\!/ \Gamma' \Rightarrow \Delta' /\!\!/ \Sigma', A \Rightarrow \Pi'$. Hence, starting from such leaves and applying the same sequence of rules as in the modal block of π_1', we have a derivation π of $\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/_{\mathsf{e}} (\Sigma, A \Rightarrow \Pi; \Omega \Rightarrow \Theta)$. Thus

$$\dfrac{\overset{\pi}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/_{\mathsf{e}} (\Sigma, A \Rightarrow \Pi; \Omega \Rightarrow \Theta)} \quad \overset{\pi_1''}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/ \Omega \Rightarrow A, \Theta}}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/_{\mathsf{e}} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)}\ \mathsf{C}$$

The other cases are similar and simpler. □

It is worth noting that modular calculi for the logics in the non-normal cube were also given in the framework of *labelled sequents* in [12, 20, 53]. The calculi presented there are very much semantically motivated. The systems in [20] are based on a translation of non-normal modal logics

$$\frac{\mathcal{G} \mathbin{/\!\!/} \Gamma \Rightarrow \Delta \mathbin{/\!\!/}_{\mathsf{m}} \Rightarrow B}{\mathcal{G} \mathbin{/\!\!/} \Gamma \Rightarrow \Box B, \Delta}\ \Box_R^{\mathsf{m}} \qquad \frac{\mathcal{G} \mathbin{/\!\!/} \Gamma \Rightarrow \Delta \mathbin{/\!\!/} \Sigma, A \Rightarrow \Pi}{\mathcal{G} \mathbin{/\!\!/} \Gamma, \Box A \Rightarrow \Delta \mathbin{/\!\!/}_{\mathsf{m}} \Sigma \Rightarrow \Pi}\ \Box_L^{\mathsf{m}} \qquad \frac{\mathcal{G} \mathbin{/\!\!/}_{\mathsf{m}} \Gamma \Rightarrow \Delta}{\mathcal{G} \mathbin{/\!\!/} \Gamma \Rightarrow \Delta}\ \mathsf{C} \qquad \frac{\mathcal{G} \mathbin{/\!\!/} \Gamma \Rightarrow \Delta}{\mathcal{G} \mathbin{/\!\!/}_{\mathsf{m}} \Gamma \Rightarrow \Delta}\ \mathsf{N}$$

$$\mathsf{LNS}_{\mathsf{M}\mathcal{A}} \quad \{\Box_R^{\mathsf{m}}, \Box_L^{\mathsf{m}}\} \cup \mathcal{A} \quad \text{for } \mathcal{A} \subseteq \{\mathsf{C}, \mathsf{N}\}$$

Fig. 9. The structural variants of the linear nested systems for monotone modal logics

into normal modal logics. The complexity of the resulting semantic conditions then is captured using *systems of rules* [52]. The calculi in [53] and in [12] avoid this translation, but introduce additional predicates in the meta-language to explicitly refer to neighbourhoods.

4 STRUCTURAL VARIANTS AND THE MODAL TESSERACT

The systems for the non-normal logics introduced in the last section make use of different *logical* rules, but sometimes it is preferable to change logics only by modifying the *structural* rules of the system, i.e., the rules governing the behaviour of the *structural connective* $/\!\!/$. In particular, for sequent systems varying the structural rules instead of the logical rules often results in higher modularity, since cut elimination proofs are usually less affected by additional structural rules. This has also been called *Došen's Principle* in [69]. We will now apply this idea to obtain modular calculi for a number of extensions of monotone modal logic M (see also [26] for a semantic treatment not only of these logics). In order to do so, we first simplify the calculus for monotone modal logic. As the avid reader might have noticed, there is quite a lot of redundancy in this calculus. In particular, after applying the rule M, the second premiss of the following applications of C or $\Box_L^{\mathsf{e}}$ become trivially derivable. Hence for the present purpose we might as well omit these premisses and the corresponding component of the nesting operator, replacing the binary operator $/\!\!/_{\mathsf{e}}$ with the unary operator $/\!\!/_{\mathsf{m}}$. Linear nested sequents for monotone modal logics then are given by:

$$\mathsf{LNS}_{\mathsf{m}} ::= \Gamma \Rightarrow \Delta \mid \Gamma \Rightarrow \Delta \mathbin{/\!\!/}_{\mathsf{m}} \Sigma \Rightarrow \Pi \mid \Gamma \Rightarrow \Delta \mathbin{/\!\!/} \mathsf{LNS}_{\mathsf{m}}$$

The rules $\Box_R^{\mathsf{e}}$ and $\Box_L^{\mathsf{e}}$ in the monotone setting then are simplified to the rules $\Box_R^{\mathsf{m}}$ and $\Box_L^{\mathsf{m}}$ of Fig. 9, which now only need to carry information about one direction of the premisses. The additional rules for the axioms C and N (shown in the same figure) now are given in their structural variants, permitting to switch from the "finished rule" marker $/\!\!/$ to the "unfinished rule" marker $/\!\!/_{\mathsf{m}}$ and back. Obviously, adding both rules N and C collapses both nesting operators into one, and essentially brings us to the linear nested sequent calculus for modal logic K from Fig. 3, as should be the case since K is precisely the logic MNC. Finally, observe that applying rule C allows propositional rules to be applied between modal phases.

The main benefit of capturing the axioms C and N by structural rules instead of logical rules is that it is now possible to give calculi for further extensions in a uniform way, independent of normality or non-normality of the base logic. The further axioms we are going to consider are (using the terminology of [26]):

$$\mathsf{P}\ \neg\Box\bot \qquad \mathsf{D}\ \neg(\Box A \wedge \Box\neg A) \qquad \mathsf{T}\ \Box A \to A \qquad 4\ \Box A \to \Box\Box A \qquad 5\ \Box A \vee \Box\neg\Box A$$

Note that we included both the two axioms P and D which are usually taken to be two different formulations of the seriality axiom. This is due to the fact that in the non-normal setting the two formulations are not equivalent: while P is derivable from D, the opposite does not hold in logics not validating the axiom C. The reason for why we here only consider extensions of monotone modal logic with these axioms instead of extensions of classical modal logic E is that obtaining cut-free sequent calculi for many of these extensions seems to be problematic [28].

$$\frac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\ \mathsf{M} \qquad \frac{\Rightarrow A}{\Rightarrow \Box A}\ \mathsf{N} \qquad \frac{A \Rightarrow}{\Box A \Rightarrow}\ \mathsf{P} \qquad \frac{A, B \Rightarrow}{\Box A, \Box B \Rightarrow}\ \mathsf{D} \qquad \frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\ \mathsf{T}$$

$$\frac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B}\ 4 \qquad \frac{\Rightarrow A, \Box B}{\Rightarrow \Box A, \Box B}\ 5 \qquad \frac{A, \Box B \Rightarrow}{\Box A, \Box B \Rightarrow}\ \mathsf{D4} \qquad \frac{A \Rightarrow \Box B}{\Box A \Rightarrow \Box B}\ \mathsf{D5}$$

$$\underset{(|\Gamma| \geq 1)}{\frac{\Gamma \Rightarrow A}{\Box \Gamma \Rightarrow \Box A}\ \mathsf{C}} \qquad \frac{\Gamma \Rightarrow}{\Box \Gamma \Rightarrow}\ \mathsf{CD} \qquad \underset{(|\Gamma, \Sigma| \geq 1)}{\frac{\Box \Gamma, \Sigma \Rightarrow B}{\Box \Gamma, \Box \Sigma \Rightarrow \Box B}\ \mathsf{C4}} \qquad \frac{\Box \Gamma, \Sigma \Rightarrow}{\Box \Gamma, \Box \Sigma \Rightarrow}\ \mathsf{CD4}$$

$$\frac{\Box \Gamma, \Sigma \Rightarrow A}{\Box \Gamma, \Box \Sigma \Rightarrow \Box A}\ \mathsf{K4} \qquad \frac{\Box \Gamma, \Sigma \Rightarrow A, \Box \Delta}{\Box \Gamma, \Box \Sigma \Rightarrow \Box A, \Box \Delta}\ \mathsf{K45} \qquad \frac{\Box \Gamma, \Sigma \Rightarrow \Box \Delta}{\Box \Gamma, \Box \Sigma \Rightarrow \Box \Delta}\ \mathsf{KD45}$$

Fig. 10. Sequent rules for extensions of monotonic logics. We slightly abuse notation and write the same letters for axioms and the corresponding rules.

$$\frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/_{\mathsf{m}} \Rightarrow}{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta}\ \mathsf{P} \qquad \frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/_{\mathsf{m}} A \Rightarrow}{\mathcal{G} /\!/ \Gamma, \Box A \Rightarrow \Delta}\ \mathsf{D} \qquad \frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/_{\mathsf{m}} \Sigma \Rightarrow \Pi}{\mathcal{G} /\!/ \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{T}$$

$$\frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ \Sigma, \Box A \Rightarrow \Pi}{\mathcal{G} /\!/ \Gamma, \Box A \Rightarrow \Delta /\!/_{\mathsf{m}} \Sigma \Rightarrow \Pi}\ 4 \qquad \frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ \Sigma \Rightarrow \Pi, \Box A}{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta, \Box A /\!/_{\mathsf{m}} \Sigma \Rightarrow \Pi}\ 5$$

$$\mathsf{LNS}_{\mathsf{M}\mathcal{A}} \quad \{\Box^{\mathsf{m}}_{R}, \Box^{\mathsf{m}}_{L}\} \cup \mathcal{A} \quad \text{for } \mathcal{A} \subseteq \{\mathsf{C}, \mathsf{N}, \mathsf{P}, \mathsf{D}, 4, 5\}$$

Fig. 11. Linear nested sequent rules for extensions of monotonic modal logics

Definition 4.1 (Sequent calculi). The sequent rules for extensions of monotone modal logic with axioms from $\{\mathsf{P}, \mathsf{D}, \mathsf{T}, 4, 5\}$ are given in Fig. 10. Let $\mathcal{A} \subseteq \{\mathsf{N}, \mathsf{P}, \mathsf{D}, \mathsf{T}, 4, 5\}$. The sequent system $\mathsf{G}_{\mathsf{M}\mathcal{A}}$ contains the standard propositional rules of G (see Def. 3.6) as well as the following modal rules:

- $\{\mathsf{M}\} \cup \mathcal{A}$
- D4 if $\{\mathsf{D}, 4\} \subseteq \mathcal{A}$
- D5 if $\{\mathsf{D}, 5\} \subseteq \mathcal{A}$.

The sequent system $\mathsf{G}_{\mathsf{MC}\mathcal{A}}$ contains the standard propositional rules with the additional rules

- $\{\mathsf{C}\} \cup \mathcal{A}$
- CD if $\mathsf{P} \in \mathcal{A}$ or $\mathsf{D} \in \mathcal{A}$
- C4 if $4 \in \mathcal{A}$
- CD4 if $\{\mathsf{P}, 4\} \subseteq \mathcal{A}$ or $\{\mathsf{D}, 4\} \subseteq \mathcal{A}$
- K4 if $\{\mathsf{N}, 4\} \subset \mathcal{A}$
- K45 if $\{\mathsf{N}, 4, 5\} \subseteq \mathcal{A}$
- KD45 if $\{\mathsf{N}, \mathsf{P}, 4, 5\} \subseteq \mathcal{A}$ or $\{\mathsf{N}, \mathsf{D}, 4, 5\} \subseteq \mathcal{A}$

Note that the modal sequent rules of Fig. 10 do not absorb weakening into the conclusion. Indeed, the weakening and contraction rules are not admissible in most of these calculi. While of course the modal rules could be modified as to make the structural rules admissible, to stay closer to the literature we consider the calculi with explicit structural rules. The additional sequent rules stipulated in the above definition are required for cut elimination. In particular, this means that modularity fails almost completely for these systems. Decomposing the rules yields the linear nested sequent rules and rule sets $\mathsf{LNS}_{\mathsf{M}\mathcal{A}}$ given in Fig. 11. Note in particular that we do not need to include additional rules at all, and hence the calculi are completely modular. As for normal modal logics, extensions of M including the axiom 5 are not as well behaved as those without it. We first consider logics not including 5. Most of the following results can be found in the literature, see the proof for the exact references.

Proposition 4.2. *For $\mathcal{A} \subseteq \{\mathsf{N}, \mathsf{C}, \mathsf{P}, \mathsf{D}, 4\}$ the sequent calculus $\mathsf{G}_{\mathsf{M}\mathcal{A}}\mathsf{ConW}$ is sound and complete for the logic $\mathsf{M}\mathcal{A}$ i.e., for every formula A we have $A \in \mathsf{M}\mathcal{A}$ if and only if $\vdash_{\mathsf{G}_{\mathsf{M}\mathcal{A}}\mathsf{ConW}} \Rightarrow A$.*

Proof. For the extensions of normal modal logic $\mathsf{K} = \mathsf{MNC}$, see e.g. [69]. For the logic MCT the result is shown in [57], for the extensions of M with axioms from $\{\mathsf{N}, \mathsf{C}\}$ see [35]. The result for the logics MP and MCP = MCD can be found in [58]. The majority of the results for the non-normal logics are due to [27], namely the calculi for all extensions of M with axioms from $\{\mathsf{N}, \mathsf{D}, \mathsf{T}, 4\}$. The remaining calculi for the logics MP, MP4, MNP, MNP4, MC4, MCP4 = MCD4 and MT4 can be constructed using methods similar to the ones in [36, 39]. The cut elimination proof for these calculi essentially is an extension of the cut elimination proof given in [27]. Since it is not central to the topic of this paper we relegate it to Appendix A. ◻

The decomposition of the rules of these sequent calculi into the linear nested sequent rules shown in Fig. 11 then provides modular systems for every combination of C, N, P, D, T, 4.

Theorem 4.3 (Soundness and Completeness). *Let $\mathcal{A}$ be a subset of $\{\mathsf{C}, \mathsf{N}, \mathsf{P}, \mathsf{D}, \mathsf{T}, 4\}$. Then the linear nested sequent calculus $\mathsf{LNS}_{\mathsf{M}\mathcal{A}}\mathsf{ConW}$ is sound and complete for the logic $\mathsf{M}\mathcal{A}$, i.e., for every formula A we have $A \in \mathsf{M}\mathcal{A}$ if and only if $\vdash_{\mathsf{LNS}_{\mathsf{M}\mathcal{A}}\mathsf{ConW}} \Rightarrow A$.*

Proof. We first show completeness by simulating sequent derivations in the last component. Here we only show how to simulate the modal rules. First, the rules $(\mathsf{M}n)$ for monotone logics including the axiom C are simulated by

$$\frac{A_1, \dots, A_n \Rightarrow B}{\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta}\,(\mathsf{M}n) \quad\rightsquigarrow\quad \dfrac{\dfrac{\dfrac{\begin{array}{c}\dfrac{\Gamma \Rightarrow \Delta \,/\!\!/\, A_1, \dots, A_n \Rightarrow B}{\Gamma, \Box A_n \Rightarrow \Delta \,/\!\!/_{\mathsf{m}} A_1, \dots, A_{n-1} \Rightarrow B}\,\Box_L^{\mathsf{m}} \\ \vdots \\ \Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta \,/\!\!/_{\mathsf{m}} A_1 \Rightarrow B\end{array}}{\Gamma, \Box A_2, \dots, \Box A_n \Rightarrow \Delta \,/\!\!/\, A_1 \Rightarrow B}\,\mathsf{C}}{\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Delta /\!\!/_{\mathsf{m}} \Rightarrow B}\,\Box_L^{\mathsf{m}}}{\Gamma, \Box A_1, \dots, \Box A_n \Rightarrow \Box B, \Delta}\,\Box_R^{\mathsf{m}}$$

Where the vertical dots abbreviate successive applications of the rules $\Box_L^{\mathsf{m}}$ and C. The case for $n = 1$ gives the simulation of the rule M. The necessitation rule N is simulated by:

$$\frac{\Rightarrow B}{\Gamma \Rightarrow \Box B, \Delta}\,\mathsf{N} \quad\rightsquigarrow\quad \dfrac{\dfrac{\Gamma \Rightarrow \Delta /\!\!/ \Rightarrow B}{\Gamma \Rightarrow \Delta /\!\!/_{\mathsf{m}} \Rightarrow B}\,\mathsf{N}}{\Gamma \Rightarrow \Box B, \Delta}\,\Box_R^{\mathsf{m}}$$

The simulations of the rules from Fig. 10 then are (omitting the general context $\mathcal{G}$):

$$\frac{A \Rightarrow}{\Box A \Rightarrow}\,\mathsf{P} \quad\rightsquigarrow\quad \dfrac{\dfrac{\Rightarrow /\!\!/ A \Rightarrow}{\Box A \Rightarrow /\!\!/_{\mathsf{m}} \Rightarrow}\,\Box_L^{\mathsf{m}}}{\Box A \Rightarrow}\,\mathsf{P} \qquad\qquad \frac{A, B \Rightarrow}{\Box A, \Box B \Rightarrow}\,\mathsf{D} \quad\rightsquigarrow\quad \dfrac{\dfrac{\Rightarrow /\!\!/ A, B \Rightarrow}{\Box A \Rightarrow /\!\!/_{\mathsf{m}} B \Rightarrow}\,\Box_L^{\mathsf{m}}}{\Box A, \Box B \Rightarrow}\,\mathsf{D}$$

$$\frac{\Gamma, A \Rightarrow \Delta}{\Gamma, \Box A \Rightarrow \Delta}\,\mathsf{T} \quad\rightsquigarrow\quad \dfrac{\dfrac{\Rightarrow /\!\!/ \Gamma, A \Rightarrow \Delta}{\Box A \Rightarrow /\!\!/_{\mathsf{m}} \Gamma \Rightarrow \Delta}\,\Box_L^{\mathsf{m}}}{\Gamma, \Box A \Rightarrow \Delta}\,\mathsf{T} \qquad\qquad \frac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B}\,4 \quad\rightsquigarrow\quad \dfrac{\dfrac{\Rightarrow /\!\!/ \Box A \Rightarrow B}{\Box A \Rightarrow /\!\!/_{\mathsf{m}} \Rightarrow B}\,4}{\Box A \Rightarrow \Box B}\,\Box_R^{\mathsf{m}}$$

$$\frac{\Box A, B \Rightarrow}{\Box A, \Box B \Rightarrow}\,\mathsf{D4} \quad\rightsquigarrow\quad \dfrac{\dfrac{\Gamma \Rightarrow \Delta /\!\!/ \Box A, B \Rightarrow}{\Gamma, \Box B \Rightarrow \Delta /\!\!/_{\mathsf{m}} B \Rightarrow}\,4}{\Gamma, \Box A, \Box B \Rightarrow \Delta}\,\mathsf{D}$$

In the presence of C we use the rule C to move additional formulae on the left hand side. E.g., for the system containing the axioms C, D and 4 we would have:

$$\frac{\Box B, A_1, A_2 \Rightarrow}{\Box B, \Box A_1, \Box A_2 \Rightarrow}\ \mathsf{CD4} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow /\!/_{\mathsf{m}} \Box B, A_1, A_2 \Rightarrow}{\Box A_2 \Rightarrow /\!/_{\mathsf{m}} \Box B, A_1 \Rightarrow}\ \Box_L^{\mathsf{m}}}{\Box A_2 \Rightarrow /\!/ \Box B, A_1 \Rightarrow}\ \mathsf{C}}{\Box B, \Box A_2 \Rightarrow /\!/_{\mathsf{m}} A_1 \Rightarrow}\ \mathsf{4}}{\Box B, \Box A_1, \Box A_2 \Rightarrow}\ \mathsf{D}$$

If the logic contains P but not D, the application of the rule D above is replaced by an application of P followed by applications of $\Box_L^{\mathsf{m}}$ and C. The cases of the rules C, CD, C4, K4 are analogous.

To show soundness we first observe that applications of the propositional rules in the last component can be permuted above blocks of applications between the rules C and $\Box_L^{\mathsf{m}}$, i.e., above blocks of rule applications where the last nesting is $/\!/_{\mathsf{m}}$. This is due to the fact that only modal rules can be applied inside the nesting $/\!/_{\mathsf{m}}$, no modal rule creates a new nesting after a $/\!/_{\mathsf{m}}$ nesting, and every modal rule keeps all the formulae occurring under the nesting $/\!/_{\mathsf{m}}$ in the conclusion at the same place. Hence we may assume that in a derivation in $\mathsf{LNS}_{\mathsf{M}\mathcal{A}}$ all the modal rules occur in a block. Then, analogously to Lemma 3.20 of the previous section, and using the formulations of Lemma 3.18 and 3.19 for the present calculi (the proofs of which are completely analogous), we convert the derivation into a derivation where all the applications of rules are end-active. Such a derivation then is converted into a sequent derivation. In particular, every modal block then can be translated into one or more modal rules in the sequent system: whenever we have a block

$$\frac{\mathcal{G} /\!/ \Gamma \Rightarrow \Delta /\!/ \Sigma \Rightarrow \Pi}{\mathcal{G} /\!/ \Gamma, \Sigma' \Rightarrow \Delta, \Pi'}$$

consisting only of modal rules, then the sequent rule

$$\frac{\Sigma \Rightarrow \Pi}{\Gamma, \Sigma' \Rightarrow \Delta, \Pi'}$$

is derivable in the corresponding sequent system. The transformations are essentially the backwards directions of the transformations given above. □

Thus we obtain modular nested sequent calculi for all the logics in what could be called the *modal tesseract* (Fig. 12), hence repairing the bridge between non-normal and normal modal logics. Note that the modal tesseract includes one side of the standard (normal) modal cube, see e.g. [6].

For logics including the axiom 5 the situation is a bit more complicated, since not all of these (in particular K5 and S5) have cut-free sequent calculi. However, while in this case we do not obtain full modularity, we still obtain calculi for a number of logics. The number of logics we need to consider in this case is greatly reduced by the following simple observation.

Lemma 4.4. *The axiom* N *is derivable in any extension of* M *including an axiom of the form* $\Box A_1 \vee \cdots \vee \Box A_n$. *In particular, the axiom* N *is derivable in* M5.

Proof. Using the fact that A_i is equivalent to $A_i \wedge \top$ for every A_i and the monotonicity axiom for $\Box$, from $\Box A_1 \vee \cdots \vee \Box A_n$ we obtain $\Box\top \vee \cdots \vee \Box\top$, which is equivalent to $\Box\top$. □

Hence the lattice of extensions of M5 with axioms from {N, C, P, D, T, 4} collapses to the 12 logics shown in Fig. 13 (the *house of* M5 – not all corners of it are safe, i.e., cut-free in the sense of sequent calculi, though). In particular, the extensions of MC5 are the same as the extensions of normal modal logic K5. Again, most of the following results are found in the literature (see the proof for the exact references).

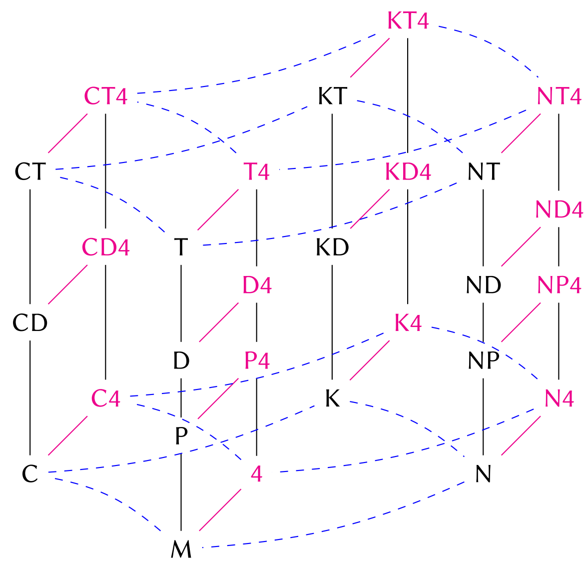


Fig. 12. The modal tesseract

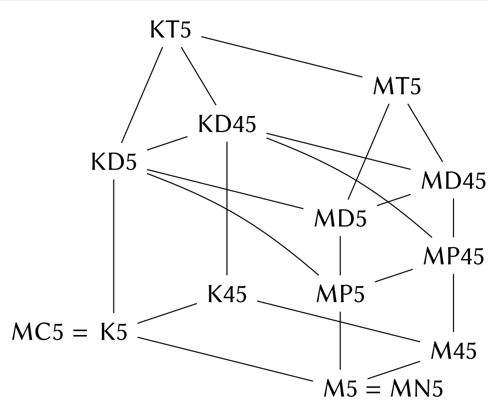


Fig. 13. The extensions of modal logic M5

Proposition 4.5. *Let $\mathcal{L}$ be one of the logics*

$$\{\mathsf{M5}, \mathsf{MP5}, \mathsf{M45}, \mathsf{MP45}, \mathsf{MD45}, \mathsf{K45}, \mathsf{KD45}\}$$

Then $\mathsf{G}_{\mathcal{L}}\mathsf{ConW}$ *is sound and complete for $\mathcal{L}$, i.e., for every formula A we have $A \in \mathcal{L}$ if and only if* $\vdash_{\mathsf{LNS}_{\mathcal{L}}} \Rightarrow A$.

Proof. For the logics K45 and KD45 this was shown in [64], but note that the calculi for K45 and KD45 considered here are slight variations of the ones in *op. cit.* In particular, there also the rule

$$\frac{\Box\Gamma, \Sigma \Rightarrow \Box\Delta}{\Box\Gamma, \Box\Sigma \Rightarrow \Box\Delta}$$

with nonempty Δ is included in the rule set for K45. This rule can be derived in the calculus considered here using the rule K45 together with a cut on the derivable sequent $\Box\Box A \Rightarrow \Box A$. Equivalence of the cut-free systems then follows from cut elimination for $\mathsf{G}_{\mathsf{K45}}$. The latter follows from the general criteria of [36, Thm. 2.3.16], but is also given explicitly in Appendix A.

In the non-normal case, for the logics M5, M45 and MD45 the result can be found in [27]. The result for the remaining two logics MP5 and MP45 follows similarly to the results of Prop. 4.2 from the cut elimination proof in Appendix A. □

For all the other cases there are counterexamples to cut elimination. In particular, for the non-normal logic MD5 this is given e.g. by the formula $\Box p \to \Box\neg\Box\neg p$. This formula is derivable using the formula $\Box p \to \neg\Box\neg p$, which is propositionally equivalent to an instance of axiom D, and the formula $\neg\Box\neg p \to \Box\neg\Box\neg p$, which is propositionally equivalent to an instance of axiom 5. However, it is not cut-free derivable in $\mathsf{G}_{\mathsf{MD5}}\mathsf{ConW}$. Interestingly, this formula is not a theorem of MP5, and hence not a counterexample to cut elimination for $\mathsf{G}_{\mathsf{MP5}}\mathsf{ConW}$.[2]

Theorem 4.6. *Let $\mathcal{L}$ be one of the logics*

$$\{\mathsf{M5}, \mathsf{MP5}, \mathsf{M45}, \mathsf{MP45}, \mathsf{MD45}, \mathsf{K45}, \mathsf{KD45}\}$$

Then $\mathsf{LNS}_{\mathcal{L}}\mathsf{ConW}$ is sound and complete for $\mathcal{L}$, i.e., for every formula A we have $A \in \mathcal{L}$ if and only if $\vdash_{\mathsf{LNS}_{\mathcal{L}}\mathsf{ConW}} \Rightarrow A$.

Proof. Analogous to the proof of Thm. 4.3. The missing transformations from sequent rules into linear nested sequent derivations are:

$$\dfrac{\Rightarrow A, \Box B}{\Rightarrow \Box A, \Box B}\ 5 \quad \rightsquigarrow \quad \dfrac{\dfrac{\Rightarrow\ \mathbin{/\!\!/}\ \Rightarrow A, \Box B}{\Rightarrow \Box B \mathbin{/\!\!/}_{\mathsf{m}} \Rightarrow A}\ 5}{\Rightarrow \Box A, \Box B}\ \Box_R^{\mathsf{m}} \qquad\qquad \dfrac{A \Rightarrow \Box B}{\Box A \Rightarrow \Box B}\ \mathsf{D5} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Rightarrow \mathbin{/\!\!/} A \Rightarrow \Box B}{\Rightarrow \Box B \mathbin{/\!\!/}_{\mathsf{m}} A \Rightarrow}\ 5}{\Box A \Rightarrow \Box B}\ \mathsf{D}$$

The rules K45 and KD45 are transformed similar to the case of CD4.

The soundness proof is analogous to the one for the cases not involving the axiom 5. □

5 RECONCILING SEQUENTS WITH NESTED SEQUENTS

We have presented a modular way of proposing several different modal systems. The beauty in this is that all systems share the same core, where modal rules can be plugged in and/or mixed together. For that, we refined sequent rules, exposing their behaviour locally. The price to pay for this modularity is, of course, efficiency, since there are more rules which could have been applied to derive a given sequent. In particular, the propositional rules could be applied in any component, giving rise to a great number of derivations which should be identified modulo bureaucracy. This alone could be taken care of by simply restricting the calculi to their end-active variants, so that the propositional rules are applied only in the last component. However, doing so would still leave open the possibility of mixing propositional and modal rules, e.g., applying (bottom-up) a rule $\Box_{iR}$ followed by a propositional rule in the last component, and then a rule $\Box_{ji_L}$. This as well is a potential source of inefficiency when compared to the sequent framework, where we have blocks of propositional rules alternating with single modal rules.

In this section, we will show how auxiliary nesting operators can be used in order to guarantee a notion of *normal form* for LNS derivations that mimic the respective sequent ones, hence reducing the proof search space and optimizing proof search.

[2] For the reader familiar with neighbourhood semantics [10, 26]: The MP5-model $(\{a, b\}, \eta, \sigma)$ with $\eta(a) = \eta(b) = \{\{a\}, \{b\}, \{a, b\}\}$ and $[\![p]\!] = \{a\}$ witnesses satisfiability of the negation of this formula.

$$\frac{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta \backslash\!\!\backslash^j \Sigma, A \Rightarrow \Pi}{\mathcal{G} /\!\!/^k \Gamma, \Box_i A \Rightarrow \Delta \backslash\!\!\backslash^j \Sigma \Rightarrow \Pi} \Box_{ij_L} \qquad \frac{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta \backslash\!\!\backslash^i \Rightarrow A}{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta, \Box_i A} \Box_{iR} \qquad \frac{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta}{\mathcal{G} \backslash\!\!\backslash^k \Gamma \Rightarrow \Delta} \text{close}$$

$$\frac{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta \backslash\!\!\backslash^j A \Rightarrow}{\mathcal{G} /\!\!/^k \Gamma, \Box_i A \Rightarrow \Delta} \mathsf{d}_{ij} \qquad \frac{\mathcal{G} /\!\!/^k \Gamma, A \Rightarrow \Delta}{\mathcal{G} /\!\!/^k \Gamma, \Box_i A \Rightarrow \Delta} \mathsf{t}_i \qquad \frac{\mathcal{G} /\!\!/^k \Gamma \Rightarrow \Delta \backslash\!\!\backslash^j \Sigma, \Box_i A \Rightarrow \Pi}{\mathcal{G} /\!\!/^k \Gamma, \Box_i A \Rightarrow \Delta \backslash\!\!\backslash^j \Sigma \Rightarrow \Pi} 4_{ij}$$

Fig. 14. Modal rules for $\mathsf{FLNS}_{(N,\preccurlyeq,F)}$, where k, i, j are as in Figure 6. The propositional rules are the same as in Fig. 1, restricted to the last component.

Definition 5.1. A LNS derivation is in *block form* if, whenever a modal rule occurs directly above a propositional rule, then that modal rule creates a new component.

In the following we use block form as the normal form of LNS derivations. Considering first the simply dependent normal multimodal logics of Section 3.1, in Fig. 14 we present $\mathsf{FLNS}_{(N,\preccurlyeq,F)}$, an end-active version for $\mathsf{LNS}_{(N,\preccurlyeq,F)}$ (Fig. 6) where all derivations are necessarily in block form: this is assured by an auxiliary nesting operator $\backslash\!\!\backslash^i$ for each $i \in N$. This operator behaves much in the same way as the "unfinished rule marker" in the systems for non-normal modal logics. However, here we explicitly include the rule close, which intuitively marks a sequent rule as finished.

This implies that, *modulo the order of application of* $\Box_{ij_L}$ *and* d_{ij} *rules*, there is a 1-1 correspondence between derivations in the end-active variant of the LNS system $\mathsf{FLNS}_{(N,\preccurlyeq,F)}\mathsf{ConW}$ and in the sequent system $\mathsf{G}_{(N,\preccurlyeq,F)}\mathsf{ConW}$ (see Fig. 5). In this way, sequent rules can be seen as *macro rules* of linear nested rules.

Since every $\mathsf{FLNS}_{(N,\preccurlyeq,F)}\mathsf{ConW}$-derivation can be translated into a $\mathsf{LNS}_{(N,\preccurlyeq,F)}\mathsf{ConW}$-derivation by replacing the nesting $\backslash\!\!\backslash^i$ everywhere by $/\!\!/^i$ and omitting every application of the rule close we immediately obtain soundness of the system $\mathsf{FLNS}_{(N,\preccurlyeq,F)}\mathsf{ConW}$. Completeness follows as mentioned above from permuting propositional and structural rules below modal rules in derivations in the end-active variant of $\mathsf{LNS}_{(N,\preccurlyeq,F)}\mathsf{ConW}$.

Observe that the *normal* modal logics presented in this paper form a particular case of simply dependent multimodal logics (with N being a singleton). Hence all derivations in the end-active variant of the correspondent $\mathsf{FLNS}_{(N,\preccurlyeq,F)}\mathsf{ConW}$ system will be in block form.

Example 5.2. The block form derivation for the normality axiom is as follows

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\overline{\cdot \Rightarrow \cdot /\!\!/ p \Rightarrow p, q}\ \text{init} \quad \overline{\cdot \Rightarrow \cdot /\!\!/ p, q \Rightarrow q}\ \text{init}}{\cdot \Rightarrow \cdot /\!\!/ p \to q, p \Rightarrow q} \to_L}{\cdot \Rightarrow \cdot \backslash\!\!\backslash\ p \to q, p \Rightarrow q} \text{close}}{\Box(p \to q), \Box p \Rightarrow \cdot \backslash\!\!\backslash\ \cdot \Rightarrow q} \Box_L}{\Box(p \to q), \Box p \Rightarrow \Box q} \Box_R}{\Rightarrow \Box(p \to q) \to (\Box p \to \Box q)} \to_R$$

All the systems presented for *non-normal* modal logics in the previous sections are end-active and in most systems the partial nesting operator already forces that all valid derivations are in block form. The exception are the systems containing the C rule (Fig. 9). In fact, this rule allows a partial nesting to begin anywhere in the derivation, not only after an application of a modal rule.

An alternative set of rules for these systems is obtained by adding the nesting operator $\backslash\!\!\backslash$ (so that the modal rules have two levels of partial processing), together with the close rule, that forces the modal block to end. We illustrate this in Fig. 15 for the system $\mathsf{FLNS}_{\mathsf{MC}}$. Again, soundness and completeness follow as above.

$$\frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta /\!\!/_{\mathsf{m}} \Rightarrow B}{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Box B, \Delta}\ \Box_R^{\mathsf{m}} \quad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta \setminus\!\!\setminus \Sigma, A \Rightarrow \Pi}{\mathcal{G} /\!\!/ \Gamma, \Box A \Rightarrow \Delta /\!\!/_{\mathsf{m}} \Sigma \Rightarrow \Pi}\ \Box_L^{\mathsf{m}} \quad \frac{\mathcal{G} /\!\!/_{\mathsf{m}} \Gamma \Rightarrow \Delta}{\mathcal{G} \setminus\!\!\setminus \Gamma \Rightarrow \Delta}\ \mathsf{Cr} \quad \frac{\mathcal{G} /\!\!/ \Gamma \Rightarrow \Delta}{\mathcal{G} \setminus\!\!\setminus \Gamma \Rightarrow \Delta}\ \mathsf{close}$$

Fig. 15. System $\mathsf{FLNS}_{\mathsf{MC}}$.

5.1 Block forms versus focused derivations

In [2], a notion of normal form for cut-free derivations in linear logic was introduced. This normal form is given by a focused proof system organised around two “phases” of proof construction: the *negative phase* for invertible inference rules and the *positive phase* for non-necessarily-invertible inference rules. Due to invertibility, when searching for a derivation it is always safe to apply, reading bottom-up, a rule for a negative formula, so these may be applied at any time. On the other hand, rules for positive formulae may require a choice or restriction on the application of rules. Hence, in the focusing discipline, negative formulae are decomposed eagerly until only positive formulae are left, then one of them is non-deterministically chosen to be focused on. Thus focused derivations alternate negative and positive phases.

Focused nested systems for modal logics were first considered in [8], where a focused variant for all modal logics of the classical S5 cube were proposed. This approach was extended to the intuitionistic case in [9].

Since block form derivations entail a notion of normal forms in LNS by alternating modal and propositional blocks, it is natural to ask if there is any relationship between focusing in nested systems and modal blocks in linear nested systems.

In (plain) nested systems, the rule $\Box_R$ is invertible (negative), since it basically implements the semantical description for the box operator, described by a *forall* connective (which has a negative behaviour). On the other hand, $\Box_L$ is a non-invertible rule (positive), since its application should be preceded by an instance of a $\Box_R$ rule.

However, in passing to linear nesting systems, the dualities of polarities for modal connectives are lost. In fact, $\Box_L$ and $\Box_R$ rules are both non-invertible in LNS_K: while the left rule can be applied only after a right rule, for the right rule a boxed formula has to be chosen in order to be processed. Hence there seems to be no natural way of polarising the modal connectives presented here.

Let us take a closer look at the sequent rule k

$$\frac{\Gamma \Rightarrow A}{\Gamma', \Box\Gamma \Rightarrow \Box A, \Delta}\ \mathsf{k}$$

and its interpretations in nested and linear nested systems.

In the *nested* system proposed in [8] all the existing right boxes can be processed in parallel (and this is invertible) and then the left boxes can be transferred one by one to all the nestings. A derivation then proceeds by running all the possible traces in parallel, and finish whenever one or more of them succeed. Although considering the box left a positive connective leads to a complete proof system, it has an inherited negative behaviour that is ignored when adopting such polarisation. Also, in the sequent rule k, the box right should be chosen, which gives it a positive behaviour, also not taken into account in the focused system proposed in [8].

In contrast, this positive/negative behaviour of box left and right rules is present in LNS_K. In fact, while $\Box_R$ is not invertible, proposing a focused version of this rule would render the resulting system incomplete. And, as mentioned before, the $\Box_L$ rule has the restriction that it can be applied only after a $\Box_R$ rule is applied, hence it has a positive behaviour. But once the new component is created by the $\Box_R$ rule, moving the left boxed formulae can be done in any order and this action is invertible, hence negative.

Thus, although modal blocks do not correspond to focusing, it produces a normal form that mimics the sequential behaviour and preserves the inherent positive/negative flavor of the box modality. Focusing, on the other side, produces normal forms that do not correspond to sequent derivations, hence the proof space is much bigger in (focused) nested systems than in (block form) linear nested systems.

6 LABELLED LINE SEQUENT SYSTEMS AND BIPOLES

A logical framework is a meta-language used for the specification of deductive systems. Embedding systems into frameworks allows for determining/analysing/proving meta-level properties of the object level specified systems. And, since logical frameworks often come with automated procedures, the meta-level machinery can often be used for proving properties of the embedded systems automatically.

Restricting our attention to logical systems, since a specific logic gives rise to specific sets of rules in different calculi, it is very important to: choose a suitable, general logical framework, able to specify a representative class of systems/calculi; and determine whether there is a general and adequate methodology for embedding deductive systems into the chosen logical framework, so that object-level properties can be uniformly proven.

In this section, we will show that linear logic (LL) [21] is a general and adequate framework for specifying a paramount subset of linear nested systems presented in this paper. This is a fundamental result which opens the possibility of exploring meta-level properties for such logical systems by extending similar results obtained for sequent systems [47–49, 56, 61].

One of the main advantages of the LNS calculi over the standard sequent calculi is that the modal operators have separate left and right rules, and that the number of principal formulae in the modal rules is bounded. While the better control on moving formulae on nested sequents facilitates the suggestion of a general method for embedding LNS systems, the locality of the rules makes the quest of proving *adequacy* of the encodings harder. In fact, determining adequate embedding maps on linear nested sequents that smoothly extend existing ones on sequents is a non trivial task, as we will show next.

We start by reformulating the LNS structure in the language of *labelled sequents* [68], using a restriction of the correspondence between nested sequents and labelled tree sequents in [23]. We then show how to use such labelled systems in order to generate bipole clauses in linear logic which adequately correspond to LNS modal rules.

6.1 Labelled systems

Let SV a countable infinite set of *state variables* (denoted by $x, y, z, \ldots$), disjoint from the set of propositional variables. A *labelled formula* has the form $x : A$ where $x \in \mathsf{SV}$ and A is a formula. If $\Gamma = \{A_1, \ldots, A_n\}$ is a multiset of formulae, then $x : \Gamma$ denotes the multiset $\{x : A_1, \ldots, x : A_n\}$ of labelled formulae. A (possibly empty) set of *relation terms* (*i.e.* terms of the form xRy, where $x, y \in \mathsf{SV}$) is called a *relation set*. For a relation set $\mathcal{R}$, the *frame* $Fr(\mathcal{R})$ defined by $\mathcal{R}$ is given by $(|\mathcal{R}|, \mathcal{R})$ where $|\mathcal{R}| = \{x \mid xRy \in \mathcal{R} \text{ or } yRx \in \mathcal{R} \text{ for some } y \in \mathsf{SV}\}$. We say that a relation set $\mathcal{R}$ is *treelike* if the frame defined by $\mathcal{R}$ is a tree or $\mathcal{R}$ is empty. A treelike relation set $\mathcal{R}$ is called *linelike* if each node in $\mathcal{R}$ has at most one child.

Definition 6.1. A *labelled line sequent* LLS is a labelled sequent $\mathcal{R}, X \Rightarrow Y$ where

(1) $\mathcal{R}$ is linelike;
(2) if $\mathcal{R} = \emptyset$ then X has the form $x_0 : A_1, \ldots, x_0 : A_n$ and Y has the form $x_0 : B_1, \ldots, x_0 : B_m$ for some $x_0 \in \mathsf{SV}$;
(3) if $\mathcal{R} \neq \emptyset$ then every state variable x that occurs in either X or Y also occurs in $\mathcal{R}$.

$$\frac{}{zRx, X, x:p \Rightarrow x:p, Y}\ \text{init} \quad \frac{}{zRx, X, x:\bot \Rightarrow Y}\ \bot_L \quad \frac{}{zRx, X \Rightarrow x:\top, Y}\ \top_R$$

$$\frac{zRx, X \Rightarrow Y, x:A}{zRx, X, x:\neg A, \Rightarrow Y}\ \neg_L \quad \frac{zRx, X, x:A, \Rightarrow Y}{zRx, X \Rightarrow Y, x:\neg A}\ \neg_R \quad \frac{zRx, X, x:A \Rightarrow Y \quad zRx, X, x:B \Rightarrow Y}{zRx, X, x:A \vee B \Rightarrow Y}\ \vee_L$$

$$\frac{zRx, X \Rightarrow Y, x:A, x:B}{zRx, X \Rightarrow Y, x:A \vee B}\ \vee_R \quad \frac{zRx, X, x:A, x:B \Rightarrow Y}{zRx, X, x:A \wedge B \Rightarrow Y}\ \wedge_L \quad \frac{zRx, X \Rightarrow Y, x:A \quad zRx, X \Rightarrow Y, x:B}{zRx, X \Rightarrow Y, x:A \wedge B}\ \wedge_R$$

$$\frac{zRx, X \Rightarrow Y, x:A \quad zRx, X, x:B \Rightarrow Y}{zRx, X, x:A \rightarrow B \Rightarrow Y}\ \rightarrow_L \quad \frac{zRx, X, x:A \Rightarrow Y, x:B}{zRx, X \Rightarrow Y, x:A \rightarrow B}\ \rightarrow_R$$

Fig. 16. The end-active version of $\mathsf{LLS_G}$. In rule init, p is atomic.

A *labelled line sequent calculus* is a labelled sequent calculus whose initial sequents and inference rules are constructed from LLS.

Observe that, in LLS, if $xRy \in \mathcal{R}$ then $uRy \notin \mathcal{R}$ and $xRv \notin \mathcal{R}$ for any $u, v \in \mathsf{SV}$ such that $u \neq x$ and $v \neq y$.

Since linear nested sequents form a particular case of nested sequents, the algorithm given in [23] can be used for generating LLS from LNS, and vice versa. However, one has to keep the linearity property invariant through inference rules. For example, the following labelled sequent rule

$$\frac{\mathcal{R}, xRy, X \Rightarrow Y, y:A}{\mathcal{R}, X, \Rightarrow Y, x:\Box A}\ \Box_R'$$

where y is fresh, is not adequate w.r.t. the system $\mathsf{LNS_K}$, since there may exist $z \in |\mathcal{R}|$ such that $xRz \in \mathcal{R}$. That is, for labelled sequents in general, freshness alone is not enough for guaranteeing unicity of x in $\mathcal{R}$. And it does not seem to be trivial to assure this unicity by using logical rules without side conditions. To avoid this problem, we slightly modify the framework by restricting $\mathcal{R}$ to singletons, that is, $\mathcal{R} = \{xRy\}$ will record only the two last components, in this case labelled by x and y, and by adding a base case $\mathcal{R} = \{x_0Rx_1\}$ for x_0, x_1 different state variables when there are no nested components. The rule for introducing $\Box_R$ then is

$$\frac{xRy, X \Rightarrow Y, y:A}{zRx, X, \Rightarrow Y, x:\Box A}\ \Box_R$$

with y fresh. Note that this solution corresponds to recording the history of the proof search up to the last two steps similar to what is outlined in [60], hence we are adopting an end-active version of LLS.

Definition 6.2. An *end-active* LLS is a singleton relation set $\mathcal{R}$ together with a sequent $X \Rightarrow Y$ of labelled formulae, written $\mathcal{R}, X \Rightarrow Y$. The rules of an *end-active* LLS *calculus* are constructed from end-active labelled line sequents such that the active formulae in a premiss $xRy, X \Rightarrow Y$ are labelled with y and the labels of all active formulae in the conclusion are in its relation set.

From now on, we will use the end-active version of the propositional rules (see Fig. 16).

We will now show how to automatically generate LLS from LNS. This is possible since the key property of end-active LNS calculi is that rules can only move formulae "forward", that is, either an active formula produces other formulae in the same component or in the next one.

Definition 6.3. For a state variable x, define the mapping $\mathbb{TL}_x$ from LNS to end-active LLS as follows

$$\begin{array}{lcll} \mathbb{TL}_{x_1}(\Gamma_1 \Rightarrow \Delta_1) & = & x_0Rx_1, x_1:\Gamma_1 \Rightarrow x_1:\Delta_1 & \\ \mathbb{TL}_{x_n}(\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \ldots /\!\!/ \Gamma_n \Rightarrow \Delta_n) & = & x_{n-1}Rx_n, x_1:\Gamma_1, \ldots, x_n:\Gamma_n \Rightarrow x_1:\Delta_1, \ldots, x_n:\Delta_n & n > 0 \end{array}$$

$$\frac{xRy, X, y{:}A \Rightarrow Y}{xRy, X, x{:}\Box A \Rightarrow Y}\ \mathbb{TL}_x(\Box_L) \qquad \frac{xRy, X \Rightarrow Y, y{:}A}{zRx, X \Rightarrow Y, x{:}\Box A}\ \mathbb{TL}_x(\Box_R)$$

Fig. 17. The modal rules of $\mathsf{LLS_K}$. The variable y in rule $\Box_R$ is fresh.

with all state variables pairwise distinct.

We can use $\mathbb{TL}_x$ in order to construct a LLS inference rule from an inference rule of an end-active LNS calculus. The procedure, that can also be automatised, is the same as the one presented in [23], as we shall illustrate next.

Example 6.4. Consider the following application of the rule $\Box_R$ of Fig. 3

$$\frac{\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \dots /\!\!/ \Gamma_{n-1} \Rightarrow \Delta_{n-1} /\!\!/ \Gamma_n \Rightarrow \Delta_n /\!\!/ \ \Rightarrow A}{\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \dots /\!\!/ \Gamma_{n-1} \Rightarrow \Delta_{n-1} /\!\!/ \Gamma_n \Rightarrow \Delta_n, \Box A}\ \Box_R$$

Applying $\mathbb{TL}_x$ to the conclusion we obtain $x_{n-1}Rx_n, X \Rightarrow Y, x_n{:}\Box A$, where $X = x_1{:}\Gamma_1, \dots, x_n{:}\Gamma_n$ and $Y = x_1{:}\Delta_1, \dots, x_n{:}\Delta_n$. Applying $\mathbb{TL}_x$ to the premise we obtain $x_nRx_{n+1}, X \Rightarrow Y, x_{n+1}{:}A$. We thus obtain an application of the LLS rule

$$\frac{x_nRx_{n+1}, X \Rightarrow Y, x_{n+1}{:}A}{x_{n-1}Rx_n, X \Rightarrow Y, x_n{:}\Box A}\ \mathbb{TL}_x(\Box_R)$$

Fig. 17 presents the end-active labelled line sequent calculus $\mathsf{LLS_K}$ for K.

The following result follows readily by transforming derivations bottom-up.

THEOREM 6.5. *$\Gamma \Rightarrow \Delta$ is provable in a certain end-active* LNS *calculus if and only if* $\mathbb{TL}_{x_1}(\Gamma \Rightarrow \Delta)$ *is provable in the corresponding end-active* LLS *calculus.*

Note that, in an end-active LLS, state variables might occur in the sequent and not in the relation set. Such formulae will remain inactive towards the leaves of the derivation and absorbed by the initial sequents in systems where weakening is admissible.

The concepts of LLS and $\mathbb{TL}_x$ can be extended in order to handle the extensions $\mathsf{LNS_m}$ and $\mathsf{LNS_e}$ by adding the relations $R_\mathsf{m} \subseteq \mathsf{SV} \times \mathsf{SV}$ and $R_\mathsf{e} \subseteq \mathsf{SV} \times (\mathsf{SV} \times \mathsf{SV})$, respectively, and defining

$$\mathbb{TL}^\mathsf{m}_{x_n}(\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \dots /\!\!/_\mathsf{m} \Gamma_n \Rightarrow \Delta_n) \quad = \quad x_{n-1}R_\mathsf{m}x_n, x_1{:}\Gamma_1, \dots, x_n{:}\Gamma_n \Rightarrow x_1{:}\Delta_1, \dots, x_n{:}\Delta_n$$

$$\mathbb{TL}^\mathsf{e}_{x_n}(\Gamma_1 \Rightarrow \Delta_1 /\!\!/ \dots /\!\!/_\mathsf{e} (\Sigma \Rightarrow \Pi; \Omega \Rightarrow \Theta)) \quad = \quad x_{n-1}R_\mathsf{e}(x_n, y_n), x_1{:}\Gamma_1, \dots, x_n{:}\Sigma, y_n{:}\Omega \Rightarrow x_1{:}\Delta_1, \dots, x_n{:}\Pi, y_n{:}\Theta$$

The corresponding LLS rules for these systems are depicted in Figs. 18, 19 and 20. Observe that this is a generalisation of the algorithm in [23], with the careful remark that, in the case of non-normal systems, the algorithm generates premisses that are weakened w.r.t. the ones presented in Fig. 18. Thus, Theorem 6.5 is also valid for all the $\mathsf{LLS_m}$ and $\mathsf{LLS_e}$ systems presented in this work. Finally, it is worth noticing that the definition of the mapping $\mathbb{TL}_x$ for auxiliary nesting operators is the same as the respective final nesting operators.

6.2 Bipoles

In this section we exploit the above mentioned fact that LNS systems often have separate left and right introduction rules for modalities in order to present a systematic way of representing labelled line nested rules as *bipole clauses*. For that, we will use (focused) linear logic (LLF), not only because it extends the works in, e.g., [49, 56], but also since this is the basis for using the rich linear logic meta-level theory in order to reason about the specified systems. It is worth noticing,

$$\frac{x_{n-1}Rx_n, X, x_n : A \Rightarrow Y \quad x_{n-1}Ry_n, X \Rightarrow Y, y_n : A}{x_{n-1}R_{\mathsf{e}}(x_n, y_n), X, x_{n-1} : \Box A \Rightarrow Y}\ \mathbb{TL}^{\mathsf{e}}_x(\Box^{\mathsf{e}}_L)$$

$$\frac{x_n R_{\mathsf{e}}(x_{n+1}, y_{n+1}), X, y_{n+1} : B \Rightarrow Y, x_{n+1} : B}{x_{n-1}Rx_n, X \Rightarrow Y, x_n : \Box B}\ \mathbb{TL}^{\mathsf{e}}_x(\Box^{\mathsf{e}}_R) \qquad \frac{x_n Rx_{n+1}, X \Rightarrow Y, x_{n+1} : B}{x_{n-1}Rx_n, X \Rightarrow Y, x_n : \Box B}\ \mathbb{TL}^{\mathsf{e}}_x(\mathsf{N})$$

$$\frac{x_{n-1}R_{\mathsf{e}}(x_n, y_n), X, y_n : \bot \Rightarrow Y}{x_{n-1}R_{\mathsf{e}}(x_n, y_n), X \Rightarrow Y}\ \mathbb{TL}^{\mathsf{e}}_x(\mathsf{M})$$

$$\frac{x_{n-1}R_{\mathsf{e}}(x_n, y_n), X, x_n : A \Rightarrow Y \quad x_{n-1}Ry_n, X \Rightarrow Y, y_n : A}{x_{n-1}R_{\mathsf{e}}(x_n, y_n), X, x_{n-1} : \Box A \Rightarrow Y}\ \mathbb{TL}^{\mathsf{e}}_x(\mathsf{C})$$

Fig. 18. System $\mathsf{LLS_e}$ for non-normal labelled systems

$$\frac{x_n R_{\mathsf{m}} x_{n+1}, X \Rightarrow Y, x_{n+1} : B}{x_{n-1}Rx_n, X \Rightarrow Y, x_n : \Box B}\ \mathbb{TL}^{\mathsf{m}}_x(\Box^{\mathsf{m}}_R) \qquad \frac{x_{n-1}Rx_n, X, x_n : B \Rightarrow Y}{x_{n-1}R_{\mathsf{m}}x_n, X, x_{n-1} : \Box B \Rightarrow Y}\ \mathbb{TL}^{\mathsf{m}}_x(\Box^{\mathsf{m}}_L)$$

$$\frac{x_{n-1}R_{\mathsf{m}}x_n, X \Rightarrow Y}{x_{n-1}Rx_n, X \Rightarrow Y}\ \mathbb{TL}^{\mathsf{m}}_x(\mathsf{C}) \qquad \frac{x_{n-1}Rx_n, X \Rightarrow Y}{x_{n-1}R_{\mathsf{m}}x_n, X \Rightarrow Y}\ \mathbb{TL}^{\mathsf{m}}_x(\mathsf{N})$$

Fig. 19. $\mathsf{LLS_m}$ for monotone labelled systems

$$\frac{x_n R_{\mathsf{m}} x_{n+1}, X \Rightarrow Y}{x_{n-1}Rx_n, X \Rightarrow Y}\ \mathbb{TL}^{\mathsf{m}}_x(\mathsf{P}) \qquad \frac{x_n Rx_{n+1}, X, x_{n+1} : \Sigma \Rightarrow Y, x_{n+1} : \Pi}{x_{n-1}R_{\mathsf{m}}x_n, X, x_n : \Sigma \Rightarrow Y, x_n : \Pi}\ \mathbb{TL}^{\mathsf{m}}_x(\mathsf{T})$$

Fig. 20. Labelled systems for extensions of monotonic modal logics

though, that our approach is general enough for specifying inference rules in other frameworks, like LKF ([42, 50]).

The set of *formulae* of LLF is given by the following grammar:

$$F ::= p \mid p^{\perp} \mid 1 \mid 0 \mid \top \mid \bot \mid F_1 \otimes F_2 \mid F_1 ⅋ F_2 \mid F_1 \,\&\, F_2 \mid F_1 \oplus F_2 \mid \exists x.F \mid \forall x.F \mid ?F \mid !F$$

The connectives $\bot, \top, \&, ⅋, \forall, ?$ are taken to be *negative*, the connectives $1, 0, \otimes, \oplus, \exists, !$ are considered to be *positive*. The notions of negative and positive polarities are extended to formulae in the natural way by considering the outermost connective. Formulae are taken to be in *negation normal form* using the standard classical linear logic dualities, e.g., $(A \otimes B)^{\perp} \equiv A^{\perp} ⅋ B^{\perp}$. Sequents in (one-sided) linear logic are multisets of linear logic formulae. *Focused Linear Logic* LLF then adds a focusing mechanism to this structure (see Sec. 5.1). We refer the reader to [21] for the rules of unfocused linear logic and to [2, 49] for the focused versions.

6.2.1 Specifying sequents. We briefly recapitulate the basic concepts of the specification of sequent-style calculi in LLF from [49]. Let *obj* be the type of object-level formulae and let $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ be two meta-level predicates on these, i.e., both of type $obj \to o$, where o is a primitive type denoting formulas. Object-level sequents of the form $B_1, \ldots, B_n \Rightarrow C_1, \ldots, C_m$ (where $n, m \geq 0$) are specified as the multiset $\lfloor B_1 \rfloor, \ldots, \lfloor B_n \rfloor, \lceil C_1 \rceil, \ldots, \lceil C_m \rceil$ within the LLF proof system. The $\lfloor \cdot \rfloor$ and $\lceil \cdot \rceil$ predicates identify which object-level formulas appear on which side of the sequent – brackets down for left (useful mnemonic: $\lfloor$ for "left") and brackets up for right. Finally, a binary relation R is specified by a meta-level atomic formula of the form $R(\cdot, \cdot)$.

6.2.2 Specifying inference sequent rules. Inference rules are specified by a re-writing clause that replaces the active formulae in the conclusion by the active formulae in the premises. The linear logic connectives indicate how these object level formulae are connected: contexts are copied (&) or split ($\otimes$), in different inference rules ($\oplus$) or in the same sequent ($\invamp$). For example, the specification of the rules of $\mathsf{LLS_K}$ (Fig. 17) is

$$\begin{array}{lll}(\Box_R) & \lceil x:\Box A\rceil^{\perp}\otimes R(z,x)^{\perp} & \otimes\, \forall y.(\lceil y:A\rceil \invamp R(x,y))\\ (\Box_L) & \lfloor x:\Box A\rfloor^{\perp}\otimes R(x,y)^{\perp} & \otimes\, \lfloor y:A\rfloor \invamp R(x,y)\end{array}$$

where all the variables are bounded by an outermost existential quantifier.

The correspondence between focusing on a formula and an induced big-step inference rule is particularly interesting when the focused formula is a *bipole*.

Definition 6.6. A *monopole* formula is a linear logic formula that is built up from atoms and occurrences of the negative connectives, with the restriction that ? has atomic scope. A *bipole* is a positive formula built from monopoles and negated atoms using only positive connectives, with the additional restriction that ! can only be applied to a monopole.

Roughly speaking, bipoles are positive formulae in which no positive connective can be in the scope of a negative one. Focusing on such a formula will produce a single positive and a single negative phase. This two-phase decomposition enables the adequate capturing of the application of an object-level inference rule by the meta-level logic. For example, focusing on the bipole clause $(\Box_R)$ will produce the derivation

$$\dfrac{\pi_1 \qquad \pi_2 \qquad \dfrac{\Psi;\Delta',\lceil y:A\rceil, R(x,y))\Uparrow}{\Psi;\Delta'\Downarrow \forall y.(\lceil y:A\rceil\invamp R(x,y))}\,[R\Uparrow,\forall,\invamp]}{\Psi;\Delta\Downarrow \exists A,x,z.\lceil x:\Box A\rceil^{\perp}\otimes R(z,x)^{\perp}\otimes\forall y.(\lceil y:A\rceil\invamp R(x,y))}\,[\exists,\otimes]$$

where $\Delta = \lceil x:\Box A\rceil \cup R(z,x)\cup\Delta'$, and π_1 and π_2 are, respectively,

$$\dfrac{}{\Psi;\lceil x:\Box A\rceil\Downarrow\lceil x:\Box A\rceil^{\perp}}\,I_1 \qquad\qquad \dfrac{}{\Psi;R(z,x)\Downarrow R(z,x)^{\perp}}\,[\exists,I_1]$$

This one-step focused derivation will: (a) consume $\lceil x:\Box A\rceil$ and $R(z,x)$; (b) create a fresh label y; and (c) add $\lceil y:A\rceil$ and $R(x,y)$ to the context. Observe that this matches *exactly* the application of the object-level rule $\mathbb{TL}_x(\Box_R)$.

When specifying a system (logical, computational, etc) into a meta-level framework, it is desirable and often mandatory that the specification is *faithful*, that is, one step of computation on the object level should correspond to one step of logical reasoning in the meta level. This is what is called *adequacy* [55].

Definition 6.7. A specification of an object sequent system is *adequate* if provability is preserved for (open) derivations, such as inference rules themselves.

Clearly not every sequent rule can be (adequately) specified in LLF. As an example, the rule $\mathbb{TL}_x^{\mathsf{m}}(\mathsf{T})$ (Fig. 20) cannot be properly specified in our setting, since it lacks a principal formula. All other LLS rules derived from LNS systems presented in this paper can be adequately specified.

As an example, Fig. 21 shows adequate specifications in LLF of the labelled systems for the logic EC. These specifications can be used for automatic proof search as illustrated by the following theorem which is shown readily using the methods in [49].

Theorem 6.8. *Let L be a* LLS *system. A sequent $\mathcal{R},\Gamma\Rightarrow\Delta$ is provable in L if and only if there is a finite $L_0\subseteq L$ with $\mathcal{L}_0$ the theory given by the clauses of an adequate specification of the inference rules of L_0 such that $\mathcal{L}_0;\mathcal{R}\Uparrow\lfloor\Gamma\rfloor,\lceil\Delta\rceil$ is provable in LLF.*

$$(\Box_R^{\mathsf{e}}) \quad \lceil x : \Box B \rceil^{\perp} \otimes R(w,x)^{\perp} \otimes \forall y \forall z.(\lceil y : B \rceil \parr \lfloor z : B \rfloor \parr R_{\mathsf{e}}(x,(y,z)))$$

$$(\Box_L^{\mathsf{e}}) \quad \lfloor x : \Box A \rfloor^{\perp} \otimes R_{\mathsf{e}}(x,(y,z))^{\perp} \otimes (\lfloor y : A \rfloor \parr R(x,y)) \otimes (\lceil z : A \rceil \parr R(x,z))$$

$$(\mathsf{C}) \quad \lfloor x : \Box A \rfloor^{\perp} \otimes R_{\mathsf{e}}(x,(y,z))^{\perp} \otimes (\lfloor y : A \rfloor \parr R_{\mathsf{e}}(x,(y,z))) \otimes (\lceil z : A \rceil \parr R(x,z))$$

Fig. 21. The LLF specification of the modal rules of $\mathsf{LLS_{EC}}$ for the logic EC.

It turns out that the encoding of LNS into LL enabled the proposal of a general theorem prover. The system (called POULE for *PrOver for seqUent and Labelled systEms* – available at http://subsell.logic.at/nestLL/) has an LLF interpreter that takes specified LLS rules (LLF clauses – the theory) and sequents and outputs a proof of the sequent, if it is provable.

The prover is *parametric* in the theory, hence it profits from the modularity of the specified systems. Indeed, since the core is a prover for focused linear logic, POULE can be transformed into a specific prover for each specified logic by parametrically adding the encoded rules of the object system as theories in LL. Hence, for example, if the prover should validade theorems in K, one only has to add the encoding of the system (as a theory) to the prover.

It is well known that *generality* often implies *inefficiency*, and POULE is no exception to that. Hence we have also done a direct implementation of LNS systems in Prolog, parametric on the modal axioms, that can be found in https://logic.at/staff/lellmann/lnsprover/. We have no intention of comparing such implementations, since they are different in nature: a direct prover built from axioms is more adequate for proving *object-level theorems*, while POULE exemplifies how the encoding together with the meta-level prover based in LL are suitable for proving *meta-level properties* (here: derivability).

7 CONCLUDING REMARKS AND FUTURE WORK

Following [43], in [37] linear nested sequents were considered as an alternative presentation for modal proof systems. Since locality often entails modularity, this enabled modular presentations for different modal systems just by adding the local rules related to the new modalities to already defined linear nested sequent systems. In [40] we continued the programme of representing modal proof systems in LNS, including suitable extensions of K, a simply dependent bimodal logic and some standard non-normal modal logics.

In this paper, we have generalised the works *op. cit.*, presenting local systems for a family of simply dependent multimodal logics as well as a large class of non-normal modal logics. All the proposed systems were proven sound and complete w.r.t. the respective sequent systems and, as a side effect, we proved that each LNS system presented in this work could be restricted to its end-active version. This enabled a notion of normal forms for LNS derivations, narrowing the proof search space and hence allowing the proposal of more efficient local proof systems. The possibility of restricting systems to their end-active versions also entails an automatic procedure for obtaining labelled sequent versions of LNS systems. Finally, we showed that the inference rules of such labelled systems can be seen as bipoles, and hence are amenable to adequate embeddings into linear logic, which enabled the implementation of a general theorem prover, parametric in the modal axioms considered.

There are at least five future research directions that could be taken from this work.

First, following the works in [49, 56], it should be possible to use the meta-theory of linear logic to draw conclusions about the object-level LNS systems. For example, the problem of providing general procedures for guaranteeing cut admissibility for nested systems is still little understood. It turns out that the cut rule has an inherent duality: the cut formula is both a conclusion of a statement and an hypothesis of another. In sequent systems, this duality is often an invariant, being preserved throughout the cut elimination process. Developing general methods for detecting such invariants enables the use of meta-level frameworks to uniformly reason about object-level properties. In [49], bipoles and focusing were enough for both: specifying sequent systems and providing sufficient meta-level conditions for cut elimination. Here, we showed that the bipole-focusing approach can be extended for specifying nested and labelled systems. However, the meta-level characterisation of cut elimination invariants for these formalisms is still an open problem.

Second, it would be interesting to see to what extend the labels in LLS reflect the semantic models behind the studied logics. In the labelled sequent framework, Kripke's relational semantics[33] is explicitly added to sequents, so that labels correspond to *worlds* [51, 68]. In LNS instead, labels correspond to the *depth* of the nesting. Goré and Ramanayake in [23] presented a direct translation between proofs in labelled and nested systems for some normal modal logics, while Fitting in [16] showed how to relate nestings with Kripke structures for intuitionistic logic. We believe it is possible to extend these ideas for relating not only (general) frames with (classical) labelled multi-modal logics, but also neighbourhood semantics labelled systems [12] with (linear) nested systems for non-normal modal logics.

A next natural topic for investigation would be to study this proof system/semantics problem in the substructural setting. The proof theoretical problem of modalities over linear logic has been first addressed in [24], where different behaviours of the exponential connectives in linear logic were studied. We have recently [38] extended Guerrini's work in order to show how normal multi-modalities can be added to linear logic, having, as a side effect, a notion of modality that smoothly extends that of subexponentials [13]. Among the many problems that are still open in this subject, two are of particular interest: is it possible to extend substructural logics with non-normal modalities in a general way, similar to what is done in [63]? If so, which is the semantical meaning of the resulting logics? We plan to investigate these problems under the linear logic view.

From a more conceptual point of view, it would also be interesting to investigate the precise connection between the linear nested sequent framework and the framework of *dual-context* calculi as considered, e.g., in [32, 60] in a constructive setting. Since in end-active linear nested sequent calculi the applications of the rules can be restricted so that only the last two components are active, we strongly suspect that every such linear nested sequent calculus can be converted into a dual-context calculus by essentially forgetting all but the last two components. However, there are calculi in (essentially) the linear nested sequent framework which apparently cannot be restricted to being end-active, such as the calculus for modal logic KB in [59], the calculi for tense logics of linear frames in [29, 30], or the calculus for the intermediate logic LC in [34]. We leave the investigation of precise criteria for when a linear nested sequent calculus can be converted into a dual-context calculus as well as a comparison of the benefits and drawbacks of each approach for future research.

Last but not least, concerning the construction of the LNS systems themselves, a natural next step would be the investigation of general methods for obtaining such systems from cut-free sequent systems, or even directly from Hilbert-style axioms.

A PROOFS OF CUT ELIMINATION

Since the calculi include the contraction rules we follow the standard method of eliminating the *multicut* rule

$$\frac{\Gamma \Rightarrow \Delta, A^n \quad A^m, \Sigma \Rightarrow \Pi}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{Mcut}$$

(with $n, m \geq 1$) instead of the standard cut rule. As usual, since the latter is a specific instance of the multicut rule, this implies cut elimination. For the sake of exposition we deviate slightly from standard terminology in the following way.

Definition A.1. The *main formula* of an application of a propositional rule or the modal rule T from Fig. 10 is the formula occurring in the conclusion with a greater multiplicity than in any of the premisses. In particular, the *main formulae* of an application of init or $\bot_L$ are all formulae occurring in the conclusion. The *main formulae* of an application of a modal rule from Fig. 10 apart from T are all the formulae occurring in the conclusion. In an application of a structural rule, i.e., Weakening or Contraction, there are no main formulae.

Hence, e.g., the formula $\Box A$ in would be a main formula in the applications of rules D and D4 below left and centre, but not in the application of rule T below right.

$$\frac{A, B \Rightarrow}{\Box A, \Box B \Rightarrow}\ \mathsf{D} \qquad \frac{\Box A, B \Rightarrow}{\Box A, \Box B \Rightarrow}\ \mathsf{D4} \qquad \frac{\Box A, B \Rightarrow}{\Box A, \Box B \Rightarrow}\ \mathsf{T}$$

In the statement of the cut elimination theorem we write $\mathsf{G}_{\mathcal{L}}\mathsf{ConWMcut}$ for the calculus $\mathsf{G}_{\mathcal{L}}\mathsf{ConW}$ with the multicut rule Mcut.

THEOREM A.2. *Let $\mathcal{L}$ be the logic $\mathsf{M}\mathcal{A}$ with $\mathcal{A} \subseteq \{\mathsf{N}, \mathsf{C}, \mathsf{P}, \mathsf{D}, 4\}$ or one of the logics in*

$$\{\mathsf{M5}, \mathsf{MP5}, \mathsf{M45}, \mathsf{MP45}, \mathsf{MD45}, \mathsf{K45}, \mathsf{KD45}\}\ .$$

Then every derivation in $\mathsf{G}_{\mathcal{L}}\mathsf{ConWMcut}$ can be converted into a derivation in $\mathsf{G}_{\mathcal{L}}\mathsf{ConW}$ with the same endsequent.

PROOF. The proof of elimination of multicut is reasonably standard by induction on the tuples (c, d) in the lexicographic ordering $<_{\mathsf{lex}}$, where c is the *complexity* of the application of multicut, i.e., the number of symbols in the cut formula, and d is its *depth*, i.e., the sum of the depths of the derivations of the two premisses of the application of multicut.

So take a topmost application

$$\dfrac{\dfrac{\begin{matrix}\mathcal{D}_1\\ \vdots\end{matrix}}{\Gamma \Rightarrow \Delta, A^n}\ R_1 \qquad \dfrac{\begin{matrix}\mathcal{D}_2\\ \vdots\end{matrix}}{A^m, \Sigma \Rightarrow \Pi}\ R_2}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\ \mathsf{Mcut}$$

of multicut in a derivation in $\mathsf{G}_{\mathcal{L}}\mathsf{ConWMcut}$. Assume that this application of multicut is of complexity c and depth d, that $\mathcal{D}_1$ and $\mathcal{D}_2$ are the derivations of the two premisses of this application, and that R_1 and R_2 are the two last applied rules in $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively. Furthermore, assume that we have shown the statement for applications of multicut with complexity c' and depth d' such that $(c', d') <_{\mathsf{lex}} (c, d)$.

If $d = 0$, then both R_1 and R_2 are one of the rules init or $\bot_L$ and the conclusion of the multicut is obtained directly by one of these rules.

So suppose $d > 0$. We distinguish cases according to whether an occurrence of the cut formula was a main formula in the last applied rule in $\mathcal{D}_1$ and $\mathcal{D}_2$ respectively.

(1) No occurrence of the cut formula is a main formula in R_1. In this case R_1 is a structural rule, the rule T, or a propositional rule apart from init, $\bot_L$. This case is handled as usual by pushing the multicut into the premiss(es) of R_1 and applying the induction hypothesis on the depth of the application of multicut. E.g., if R_1 is $\vee_L$, the derivation $\mathcal{D}_1$ ends in

$$\frac{\Gamma', B \Rightarrow \Delta, A^n \qquad \Gamma', C \Rightarrow \Delta, A^n}{\Gamma', B \vee C \Rightarrow \Delta, A^n}\ \vee_L$$

From this we obtain a new derivation ending in

$$\dfrac{\dfrac{\Gamma', B \Rightarrow \Delta, A^n \qquad \begin{matrix}\mathcal{D}_2\\ \vdots \\ A^m, \Sigma \Rightarrow \Pi\end{matrix}}{\Gamma', B, \Sigma \Rightarrow \Delta, \Pi}\;\mathsf{Mcut} \qquad \dfrac{\Gamma', C \Rightarrow \Delta, A^n \qquad \begin{matrix}\mathcal{D}_2\\ \vdots \\ A^m, \Sigma \Rightarrow \Pi\end{matrix}}{\Gamma', C, \Sigma \Rightarrow \Delta, \Pi}\;\mathsf{Mcut}}{\Gamma', B \vee C, \Sigma, \Sigma \Rightarrow \Delta, \Pi}\;\vee_L$$

Now the two applications of multicut have complexity c and depth less than d and we are done using the induction hypothesis.

(2) At least one occurrence of the cut formula is a main formula in R_1, but none of its occurrences is a main formula in R_2. This case is analogous to the previous case, but pushing the multicut into the premiss(es) of R_2 instead of R_1.

(3) Some occurrences of the cut formula are main formulae both in R_1 and R_2. In this case we have $c > 1$, since for $c = 1$ the cut formula A is a propositional variable or $\bot$, and since some of its occurrences are main formulae both in R_1 and R_2 we would have $d = 0$. So the rules R_1, R_2 must be propositional rules apart from init, $\bot_L$ or modal rules. As usual we distinguish cases according to the last applied rules R_1, R_2, and first apply *cross-cuts*, i.e., multicuts on the premiss(es) of R_1 and the conclusion of R_2 and vice versa to eliminate occurrences of the cut formula from the premisses of the two rules. These multicuts then have smaller depth and are eliminated using the induction hypothesis. Then we reduce the complexity of the multicut. Since the propositional cases are standard we only treat an exemplary case.

(a) $R_1 = \vee_R$ and $R_2 = \vee_L$. Then the derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ end in

$$\dfrac{\Gamma \Rightarrow \Delta, B \vee C^{n-1}, B, C}{\Gamma \Rightarrow \Delta, B \vee C^n}\;\vee_R \qquad \dfrac{B, B \vee C^{m-1}, \Sigma \Rightarrow \Pi \qquad C, B \vee C^{m-1}, \Sigma \Rightarrow \Pi}{B \vee C^m, \Sigma \Rightarrow \Pi}\;\vee_L$$

From this we obtain derivations ending in

$$\dfrac{\Gamma \Rightarrow \Delta, B \vee C^{n-1}, B, C \qquad \dfrac{B, B \vee C^{m-1}, \Sigma \Rightarrow \Pi \quad C, B \vee C^{m-1}, \Sigma \Rightarrow \Pi}{B \vee C^m, \Sigma \Rightarrow \Pi}\;\vee_L}{\Gamma, \Sigma \Rightarrow \Delta, \Pi, B, C}\;\mathsf{Mcut}$$

and

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, D \vee C^{n-1}, B, C}{\Gamma \Rightarrow \Delta, B \vee C^n}\;\vee_R \qquad D, B \vee C^{m-1}, \Sigma \Rightarrow \Pi}{D, \Gamma, \Sigma \Rightarrow \Delta, \Pi}\;\mathsf{Mcut}$$

with D either of the formulae C, D. These two applications of multicut have complexity c and depth less than d and hence are eliminated using the induction hypothesis. From the resulting derivations finally we obtain a derivation ending in

$$\dfrac{\dfrac{\dfrac{\Gamma, \Sigma \Rightarrow \Delta, \Pi, B, C \qquad B, \Gamma, \Sigma \Rightarrow \Delta, \Pi}{\Gamma, \Sigma, \Gamma, \Sigma \Rightarrow \Delta, \Pi, \Delta, \Pi, C}\;\mathsf{Mcut} \qquad C, \Gamma, \Sigma \Rightarrow \Delta, \Pi}{\Gamma, \Sigma, \Gamma, \Sigma, \Gamma, \Sigma \Rightarrow \Delta, \Pi, \Delta, \Pi, \Delta, \Pi}\;\mathsf{Mcut}}{\Gamma, \Sigma \Rightarrow \Delta, \Pi}\;\mathsf{Con}$$

Here the newly introduced applications of Mcut have depth possibly greater than d but complexity less than c, and hence also are eliminated using the induction hypothesis.

(b) $R_1 = \mathsf{M}$: In this case the logic is $\mathsf{M}\mathcal{A}$ for $\mathcal{A} \subseteq \{\mathsf{N}, \mathsf{P}, \mathsf{D}, 4\}$ or one of $\{\mathsf{M5}, \mathsf{MP5}, \mathsf{M45}, \mathsf{MP45}, \mathsf{MD45}\}$. So R_2 is one of M, P, D, T, 4, D4, D5. For the sake of brevity, in the following we only show the reductions of the multicuts, denoted by $\rightsquigarrow$. As usual, the newly introduced multicuts have the same complexity but lower depth, or of lower complexity than the original ones.

(i) $R_2 = (\text{M})$:

$$\dfrac{\dfrac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\,\text{M} \quad \dfrac{B \Rightarrow C}{\Box B \Rightarrow \Box C}\,\text{M}}{\Box A \Rightarrow \Box C}\,\text{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{A \Rightarrow B \quad B \Rightarrow C}{A \Rightarrow C}\,\text{Mcut}}{\Box A \Rightarrow \Box C}\,\text{M}$$

(ii) $R_2 = \text{P}$: similar to the previous case.

(iii) $R_2 = \text{D}$: The case where $m = 1$ is as above. If $m > 1$ we only need to add some structural rules:

$$\dfrac{\dfrac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\,\text{M} \quad \dfrac{B, B \Rightarrow}{\Box B, \Box B \Rightarrow}\,\text{D}}{\Box A \Rightarrow}\,\text{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{A \Rightarrow B \quad B, B \Rightarrow}{A \Rightarrow}\,\text{Mcut}}{A, A \Rightarrow}\,\text{W}}{\Box A, \Box A \Rightarrow}\,\text{D}}{\Box A \Rightarrow}\,\text{Con}$$

(iv) $R_2 = \text{T}$:

$$\dfrac{\dfrac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\,\text{M} \quad \dfrac{\Gamma, \Box B^{m-1}, B \Rightarrow \Delta}{\Gamma, \Box B^{m} \Rightarrow \Delta}\,\text{T}}{\Gamma, \Box A \Rightarrow \Delta}\,\text{Mcut} \rightsquigarrow \dfrac{\dfrac{A \Rightarrow B \quad \dfrac{\dfrac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\,\text{M} \quad \Gamma, \Box B^{m-1}, B \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta}\,\text{Mcut}}{\Gamma, A \Rightarrow \Delta}\,\text{Mcut}}{\Gamma, \Box A \Rightarrow \Delta}\,\text{T}$$

(v) $R_2 = 4$:

$$\dfrac{\dfrac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\,\text{M} \quad \dfrac{\Box B \Rightarrow C}{\Box B \Rightarrow \Box C}\,4}{\Box A \Rightarrow}\,\text{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\,\text{M} \quad \Box B \Rightarrow C}{\Box A \Rightarrow C}\,\text{Mcut}}{\Box A \Rightarrow \Box C}\,4$$

(vi) $R_2 = \text{D4}$: We consider the case that $m = 2$. The other cases are as above.

$$\dfrac{\dfrac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\,\text{M} \quad \dfrac{\Box B, B \Rightarrow}{\Box B, \Box B \Rightarrow}\,\text{D4}}{\Box A \Rightarrow}\,\text{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{A \Rightarrow B \quad \dfrac{\dfrac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\,\text{M} \quad \Box B, B \Rightarrow}{\Box A, B \Rightarrow}\,\text{Mcut}}{\Box A, A \Rightarrow}\,\text{Mcut}}{\Box A, \Box A \Rightarrow}\,\text{D4}}{\Box A \Rightarrow}\,\text{Con}$$

(vii) $R_2 = \text{D5}$: as for M

(c) $R_1 = \text{N}$:

(i) $R_2 = \text{M}$: Similar to case 3(b)ii

(ii) $R_2 = \text{P}$: As in the previous case.

(iii) $R_2 = \text{D}$: The case where $m = 2$ is similar to the previous case. For $m = 1$ we have:

$$\dfrac{\dfrac{\Rightarrow A}{\Rightarrow \Box A}\,\text{N} \quad \dfrac{A, B \Rightarrow}{\Box A, \Box B \Rightarrow}\,\text{D}}{\Box B \Rightarrow}\,\text{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow A \quad A, B \Rightarrow}{B \Rightarrow}\,\text{Mcut}}{B, B \Rightarrow}\,\text{W}}{\Box B, \Box B \Rightarrow}\,\text{D}}{\Box B \Rightarrow}\,\text{Con}$$

(iv) $R_2 = \text{T}$: Similar to case 3(b)iv.

(v) $R_2 = 4$: Like case 3(b)v.

(vi) R_2 = D4: The case with $m = 2$ is like case 3(b)vi. If $m = 1$ again we need some structural rules:

$$\dfrac{\dfrac{\Rightarrow A}{\Rightarrow \Box A}\,\text{N} \quad \dfrac{\Box A, B \Rightarrow}{\Box A, \Box B \Rightarrow}\,\text{D4}}{\Box B \Rightarrow}\,\text{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow A}{\Rightarrow \Box A}\,\text{N} \quad \Box A, B \Rightarrow}{B \Rightarrow}\,\text{Mcut}}{B, \Box B \Rightarrow}\,\text{W}}{\Box B, \Box B \Rightarrow}\,\text{D4}}{\Box B \Rightarrow}\,\text{Con}$$

(vii) R_2 = D5: As for the previous case.

(viii) R_2 = C: We have the following (substituting N for C in the last step if Γ is empty):

$$\dfrac{\dfrac{\Rightarrow A}{\Rightarrow \Box A}\,\text{N} \quad \dfrac{A^m, \Gamma \Rightarrow B}{\Box A^m, \Box\Gamma \Rightarrow \Box B}\,\text{C}}{\Box\Gamma \Rightarrow \Box B}\,\text{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Rightarrow A \quad A^m, \Gamma \Rightarrow B}{\Gamma \Rightarrow B}\,\text{Mcut}}{\Box\Gamma \Rightarrow \Box B}\,\text{C}$$

(ix) R_2 = CD: As for the previous case.

(x) R_2 = C4: Similar to case 3(c)xiii.

(xi) R_2 = CD4: Similar to case 3(c)xiii.

(xii) R_2 = K4: Similar to case 3(c)xiii.

(xiii) R_2 = K45:

$$\dfrac{\dfrac{\Rightarrow A}{\Rightarrow \Box A}\,\text{N} \quad \dfrac{\Box A^{m-k}, \Box\Gamma, A^k, \Sigma \Rightarrow B, \Box\Delta}{\Box A^m, \Box\Gamma, \Box\Sigma \Rightarrow \Box B, \Box\Delta}\,\text{K45}}{\Box\Gamma, \Box\Sigma \Rightarrow \Box B, \Box\Delta}\,\text{Mcut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\Rightarrow A \quad \dfrac{\dfrac{\Rightarrow A}{\Rightarrow \Box A}\,\text{N} \quad \Box A^{m-k}, \Box\Gamma, A^k, \Sigma \Rightarrow B, \Box\Delta}{\Box\Gamma, A^k, \Sigma \Rightarrow B, \Box\Delta}\,\text{Mcut}}{\Box\Gamma, \Sigma \Rightarrow B, \Box\Delta}\,\text{Mcut}}{\Box\Gamma, \Box\Sigma \Rightarrow \Box B, \Box\Delta}\,\text{K45}$$

(xiv) R_2 = KD45: Similar to case 3(c)xiii.

(d) R_1 = 4: Since the rule 4 is a special case of each of C4, K4, and K45, here we only treat the cases not involving C, i.e., where R_2 is one of M, P, D, T, 4, D4. The case of D5 does not occur with the considered logics.

(i) R_2 = (M): similar to case 3(b)i

(ii) R_2 = P:

$$\dfrac{\dfrac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B}\,4 \quad \dfrac{B \Rightarrow}{\Box B \Rightarrow}}{\Box A \Rightarrow}\,\text{Mcut} \quad \rightsquigarrow \quad \dfrac{\Box A \Rightarrow B \quad B \Rightarrow}{\Box A \Rightarrow}\,\text{Mcut}$$

(iii) R_2 = D: The case where $m = 2$ is similar to the previous case. If $m = 1$ we have the following reduction, using the fact that whenever $4, \text{D} \in \mathcal{A}$, then $\mathsf{G}_{\mathsf{M}\mathcal{A}}$ contains the rule D4:

$$\dfrac{\dfrac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B}\,4 \quad \dfrac{B, C \Rightarrow}{\Box B, \Box C \Rightarrow}\,\text{D}}{\Box A, \Box C \Rightarrow}\,\text{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Box A \Rightarrow B \quad C, B \Rightarrow}{\Box A, C \Rightarrow}\,\text{Mcut}}{\Box A, \Box C \Rightarrow}\,\text{D4}$$

(iv) R_2 = T:

$$\dfrac{\dfrac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B}\,4 \quad \dfrac{\Gamma, \Box B^{m-1}, B \Rightarrow \Delta}{\Gamma, \Box B^m \Rightarrow \Delta}\,\text{T}}{\Gamma, \Box A \Rightarrow \Delta}\,\text{Mcut}$$

$$\rightsquigarrow \dfrac{\dfrac{\Box A \Rightarrow B \quad \dfrac{\dfrac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B}\,4 \quad \Gamma, \Box B^{m-1}, B \Rightarrow \Delta}{\Gamma, B \Rightarrow \Delta}\,\mathsf{Mcut}}{\Gamma, \Box A, \Box A \Rightarrow \Delta}\,\mathsf{Mcut}}{\Gamma, \Box A \Rightarrow \Delta}\,\mathsf{Con}$$

(v) $R_2 = 4$:

$$\dfrac{\dfrac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\,4 \quad \dfrac{\Box B \Rightarrow C}{\Box B \Rightarrow \Box C}\,4}{\Box A \Rightarrow}\,\mathsf{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{A \Rightarrow B}{\Box A \Rightarrow \Box B}\,4 \quad \Box B \Rightarrow C}{\Box A \Rightarrow C}\,\mathsf{Mcut}}{\Box A \Rightarrow \Box C}\,4$$

(vi) $R_2 = \mathsf{D4}$: The case where $m = 1$ is similar to the previous case respectively case 3(d)vi. If $m = 2$ we have (similarly to case 3(d)iv):

$$\dfrac{\dfrac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B}\,4 \quad \dfrac{B, \Box B \Rightarrow}{\Box B, \Box B \Rightarrow}\,\mathsf{D4}}{\Box A \Rightarrow}\,\mathsf{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Box A \Rightarrow B \quad \dfrac{\dfrac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B}\,4 \quad B, \Box B \Rightarrow}{\Box A, B \Rightarrow}\,\mathsf{Mcut}}{\Box A, \Box A \Rightarrow}\,\mathsf{Mcut}}{\Box A \Rightarrow}\,\mathsf{Con}$$

(e) $R_1 = 5$: Again, since the rule 5 is a special case of rule K45, here we only consider the cases not including C, i.e., where the logic is not K45 or KD45. The remaining cases are treated in case 3j. In this case R_2 then is one of M, P, D, 4, D4, D5.

(i) $R_2 = \mathsf{M}$: Similar to case 3(b)vi.

(ii) $R_2 = \mathsf{P}$: Similar to case 3(c)vi.

(iii) $R_2 = \mathsf{D}$: Where $m = 1$ this is similar to case 3(d)iii, using that in this case the calculus also includes the rules D5 and 4. Where $n = 1$ and $m = 2$, this is similar to case 3(c)vi.

(iv) $R_2 = 4$: For $n = 2$ we have:

$$\dfrac{\dfrac{\Rightarrow A, \Box A}{\Rightarrow \Box A, \Box A}\,5 \quad \dfrac{\Box A \Rightarrow B}{\Box A \Rightarrow \Box B}\,4}{\Rightarrow \Box B}\,\mathsf{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Rightarrow A, \Box A}{\Rightarrow \Box A, \Box A}\,5 \quad \Box A \Rightarrow B}{\Rightarrow B}\,\mathsf{Mcut}}{\Rightarrow B, \Box B}\,\mathsf{W}}{\Rightarrow \Box B, \Box B}\,5}{\Rightarrow \Box B}\,\mathsf{Con}$$

For $n = 1$ we have:

$$\dfrac{\dfrac{\Rightarrow A, \Box B}{\Rightarrow \Box A, \Box B}\,5 \quad \dfrac{\Box B \Rightarrow C}{\Box B \Rightarrow \Box C}\,4}{\Rightarrow \Box A, \Box C}\,\mathsf{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Rightarrow A, \Box B \quad \dfrac{\Box B \Rightarrow C}{\Box B \Rightarrow \Box C}\,4}{\Rightarrow A, \Box C}\,\mathsf{Mcut}}{\Rightarrow \Box A, \Box C}\,5$$

(v) $R_2 = \mathsf{D4}$: The case of $n = 1$ and $m = 2$ is similar to case 3(e)ii, that for $n = 2$ and $m = 1$ to case 3(c)vi. For $n = m = 2$ we have:

$$\dfrac{\dfrac{\Rightarrow A, \Box A}{\Rightarrow \Box A, \Box A}\,5 \quad \dfrac{A, \Box A \Rightarrow}{\Box A, \Box A \Rightarrow}\,\mathsf{D4}}{\Rightarrow}\,\mathsf{Mcut}$$

$$\rightsquigarrow \dfrac{\dfrac{\Rightarrow A, \Box A \quad \dfrac{A, \Box A \Rightarrow}{\Box A, \Box A \Rightarrow}\,\mathsf{D4}}{\Rightarrow A}\,\mathsf{Mcut} \quad \dfrac{\dfrac{\Rightarrow A, A}{\Rightarrow \Box A, \Box A}\,5 \quad A, \Box A \Rightarrow}{A \Rightarrow}\,\mathsf{Mcut}}{\Rightarrow}\,\mathsf{Mcut}$$

The case of $n = m = 1$ is similar to cases 3(d)iii and 3(d)v, using the fact that in this case the calculus also includes the rules 4 and D5.

(vi) R_2 = D5: Similar to the previous case.

(f) R_1 = D5: Again, we only consider the cases not including C, for the remaining case see case 3k. The only relevant cases then are that R_2 is one of M, P, D, 4, D4, D5.

(i) R_2 = M: Similar to case 3(b)v:

$$\dfrac{\dfrac{A \Rightarrow \Box B}{\Box A \Rightarrow \Box B}\,\text{D5} \qquad \dfrac{B \Rightarrow C}{\Box B \Rightarrow \Box C}\,\text{M}}{\Box A \Rightarrow \Box C}\,\text{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{A \Rightarrow \Box B \qquad \dfrac{B \Rightarrow C}{\Box B \Rightarrow \Box C}\,\text{M}}{A \Rightarrow \Box C}\,\text{Mcut}}{\Box A \Rightarrow \Box C}\,\text{D5}$$

(ii) R_2 = P: Similar the previous case.

(iii) R_2 = D: The case of $m = 2$ is similar to the previous case, with additional structural rules. The case of $m = 1$ is similar to case 3(b)v, using that in this case the calculus also includes the rule D4..

(iv) R_2 = 4: Similar to case 3(f)i.

(v) R_2 = D4: Similar to case 3(f)iii.

(vi) R_2 = D5: As in case 3(f)i.

(g) R_1 = C:

(i) R_2 = P: Similar to case 3(b)ii, using that in this case also CD is in the rule set:

$$\dfrac{\dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,\text{C} \qquad \dfrac{A \Rightarrow}{\Box A \Rightarrow}\,\text{P}}{\Box\Gamma \Rightarrow}\,\text{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma \Rightarrow A \qquad A \Rightarrow}{\Gamma \Rightarrow}\,\text{Mcut}}{\Box\Gamma \Rightarrow}\,\text{CD}$$

(ii) R_2 = D: Similar to the previous case.

(iii) R_2 = T:

$$\dfrac{\dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,\text{C} \qquad \dfrac{\Sigma, \Box A^{m-1}, A \Rightarrow \Pi}{\Sigma, \Box A^{m} \Rightarrow \Pi}\,\text{T}}{\Sigma, \Box\Gamma \Rightarrow \Pi}\,\text{Mcut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \Rightarrow A \qquad \dfrac{\dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,\text{C} \qquad \Sigma, \Box A^{m-1}, A \Rightarrow \Pi}{\Sigma, \Box\Gamma, A \Rightarrow \Pi}\,\text{Mcut}}{\Sigma, \Gamma, \Box\Gamma \Rightarrow \Pi}\,\text{Mcut}}{\Sigma, \Box\Gamma, \Box\Gamma \Rightarrow \Pi}\,\text{T}}{\Sigma, \Box\Gamma, \Rightarrow \Pi}\,\text{Con}$$

(iv) R_2 = 4: See case 3(g)xi.

(v) R_2 = D4: Similar to case 3(g)xi.

(vi) R_2 = D5: See case 3(g)xiii.

(vii) R_2 = C:

$$\dfrac{\dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,\text{C} \qquad \dfrac{A^{m}, \Sigma \Rightarrow B}{\Box A^{m}, \Box\Sigma \Rightarrow \Box B}\,\text{C}}{\Box\Gamma, \Box\Sigma \Rightarrow \Box B}\,\text{Mcut} \quad \rightsquigarrow \quad \dfrac{\dfrac{\Gamma \Rightarrow A \qquad A^{m}, \Sigma \Rightarrow B}{\Gamma, \Sigma \Rightarrow B}\,\text{Mcut}}{\Box\Gamma, \Box\Sigma \Rightarrow \Box B}\,\text{C}$$

(viii) R_2 = CD: Similar to the previous case.

(ix) R_2 = C4:

$$\dfrac{\dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,\text{C} \qquad \dfrac{\Box A^{m-k}, \Box\Sigma, A^{k}, \Omega \Rightarrow B}{\Box A^{m}, \Box\Sigma, \Box\Omega \Rightarrow \Box B}\,\text{K4}}{\Box\Gamma, \Box\Sigma, \Box\Omega \Rightarrow \Box B}\,\text{Mcut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\Gamma \Rightarrow A \qquad \dfrac{\dfrac{\Gamma \Rightarrow A}{\Box\Gamma \Rightarrow \Box A}\,\text{C} \qquad \Box A^{m-k}, \Box\Gamma, \Box\Sigma, A^k, \Omega \Rightarrow B}{\Box\Gamma, \Box\Sigma, A^k, \Omega \Rightarrow B}\,\text{Mcut}}{\Gamma, \Box\Gamma, \Box\Sigma, \Omega \Rightarrow \Box B}\,\text{Mcut}}{\Box\Gamma, \Box\Gamma, \Box\Sigma, \Box\Omega \Rightarrow \Box B}\,\text{K4}}{\Box\Gamma, \Box\Sigma, \Box\Omega \Rightarrow \Box B}\,\text{Con}$$

(x) R_2 = CD4: Similar to case 3(g)ix.
(xi) R_2 = K4: See case 3(g)ix.
(xii) R_2 = K45: Similar to case 3(g)xiii.
(xiii) R_2 = KD45: Similar to case 3(g)xi:

(h) R_1 = C4: Similar to case 3g.
(i) R_1 = K4: See case 3h and case 3c.
(j) R_1 = K45: This case only occurs for the logics K45 and KD45, limiting the possible cases to the following:
(i) R_2 = 4: See case 3(j)ix.
(ii) R_2 = D4: See case 3(j)x.
(iii) R_2 = D5: See case 3(j)x.
(iv) R_2 = C: See case 3(j)ix.
(v) R_2 = CD: See case 3(j)x.
(vi) R_2 = C4: See case 3(j)ix.
(vii) R_2 = CD4: See case 3(j)x.
(viii) R_2 = K4: See case 3(j)ix.
(ix) R_2 = K45: We show the most interesting case, the remaining cases are similar.

$$\dfrac{\dfrac{\Box\Gamma, \Sigma \Rightarrow A, \Box A^{n-1}, \Box\Delta}{\Box\Gamma, \Box\Sigma \Rightarrow \Box A^n, \Box\Delta}\,\text{K45} \qquad \dfrac{\Box A^{m-k}, \Box\Omega, A^k, \Theta \Rightarrow B, \Box\Pi}{\Box A^m, \Box\Omega, \Box\Theta \Rightarrow \Box B, \Box\Pi}\,\text{K45}}{\Box\Gamma, \Box\Sigma, \Box\Omega, \Box\Theta \Rightarrow \Box\Delta, \Box B, \Box\Pi}\,\text{Mcut}$$

$$\rightsquigarrow \quad \dfrac{\dfrac{\dfrac{\begin{matrix}\vdots\ \mathcal{D}_1\\ \Box\Gamma, \Sigma, \Box\Omega, \Box\Theta \Rightarrow A, \Box\Delta, \Box B, \Box\Pi\end{matrix} \qquad \begin{matrix}\vdots\ \mathcal{D}_2\\ \Box\Gamma, \Box\Sigma, \Box\Omega, A^k, \Theta \Rightarrow \Box\Delta, B, \Box\Pi\end{matrix}}{\Box\Gamma, \Sigma, \Box\Omega, \Box\Theta, \Box\Gamma, \Box\Sigma, \Box\Omega, \Theta \Rightarrow \Box\Delta, \Box B, \Box\Pi, \Box\Delta, B, \Box\Pi}\,\text{Mcut}}{\Box\Gamma, \Box\Sigma, \Box\Omega, \Box\Theta, \Box\Gamma, \Box\Sigma, \Box\Omega, \Box\Theta \Rightarrow \Box\Delta, \Box B, \Box\Pi, \Box\Delta, \Box B, \Box\Pi}\,\text{K45}}{\Box\Gamma, \Box\Sigma, \Box\Omega, \Box\Theta \Rightarrow \Box\Delta, \Box B, \Box\Pi}\,\text{Con}$$

where $\mathcal{D}_1$ is

$$\dfrac{\Box\Gamma, \Sigma \Rightarrow A, \Box\Delta \qquad \dfrac{\Box A^{m-k}, \Box\Omega, A^k, \Theta \Rightarrow B, \Box\Pi}{\Box A^m, \Box\Omega, \Box\Theta \Rightarrow \Box B, \Box\Pi}\,\text{K45}}{\Box\Gamma, \Sigma, \Box\Omega, \Box\Theta \Rightarrow A, \Box\Delta, \Box B, \Box\Pi}\,\text{Mcut}$$

and $\mathcal{D}_2$ is

$$\dfrac{\dfrac{\Box\Gamma, \Sigma \Rightarrow A, \Box A^{n-1}, \Box\Delta}{\Box\Gamma, \Box\Sigma \Rightarrow \Box A^n, \Box\Delta}\,\text{K45} \qquad \Box A^{m-k}, \Box\Omega, A^k, \Theta \Rightarrow B, \Box\Pi}{\Box\Gamma, \Box\Sigma, \Box\Omega, A^k, \Theta \Rightarrow \Box\Delta, B, \Box\Pi}\,\text{Mcut}$$

(x) R_2 = KD45: Similar to the previous case.

(k) R_1 = KD45: Similar to case 3j.

□

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