Dyadic Obligations: Proofs and Countermodels via Hypersequents

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Abstract. The basic system **E** of dyadic deontic logic proposed by Åqvist offers a simple solution to contrary-to-duty paradoxes and allows to represent norms with exceptions. We investigate **E** from a proof-theoretical viewpoint. We propose a hypersequent calculus with good properties, the most important of which is cut-elimination, and the consequent subformula property. The calculus is refined to obtain a decision procedure for **E** and an effective countermodel computation in case of failure of proof search. By means of the refined calculus, we prove that validity in **E** is Co-NP and countermodels have polynomial size.

1 Introduction

Deontic logic deals with obligation and other normative concepts, which are important in a variety of fields—from law and ethics to artificial intelligence.

Obligations are contextual in nature, and take the form of conditional statements ("if-then"). Their formal analysis rely on dyadic deontic systems. The family of those systems that come with a "preference-based" semantics is the best known one. It was originally developed by [6,11], and adapted to a modal logic setting by Åqvist [2] and Lewis [16]. The framework has roots in the so-called classical theory of rational choice, sharing the assumption that a normative judgment is based on a maximization process of normative preferences. In that framework, $\bigcirc(B/A)$ (reading: "B is obligatory, given A") is true when the best A-worlds are all B-worlds. The framework was early recognized as a landmark, due to its ability to handle at once two different kinds of deontic conditionals, whose treatment within a usual Kripke semantics had proved elusive: (a) Contrary-to-duty (CTD) conditionals, and (b) Defeasible deontic conditionals. The former are obligations that come into force when some other obligation is violated. As is well-known (e.g. [7]), deontic logicians have struggled with the the problem of giving a formal treatment to CTD obligations. According to Hansson [11], van Fraassen [29], Lewis [16] and others, the problems raised by CTDs call for an ordering on possible worlds in terms of preference (or relative goodness, or betterness), and Kripke-style models fail in as much as they do not allow for grades of ideality. The use of a preference relation has also been advocated for the analysis of defeasible conditional obligations. In particular, Alchourrón [1] argues that preferential models provide a better treatment of this notion than the usual Kripke-style models. Indeed, a defeasible conditional obligation leaves room for exceptions. Under a preference-based approach, we no longer have the deontic analogue of two laws, the failure of which constitutes the main formal feature expected from defeasible conditionals; these are "deontic" modus-ponens (or Factual Detachment): $\bigcirc(B/A)$ and A imply $\bigcirc B$, and Strengthening of the Antecedent: $\bigcirc(B/A)$ entails $\bigcirc(B/A \wedge C)$. (There is an extensive literature on the treatment of contrary-to-duties, e.g. [29,16,26,17,23], and defeasible conditional obligations, e.g. [18,3,28,13], in a preference-based framework.)

The meta-theory of the framework has been the focus of much research in recent years (for an overview, see [22]). Like in traditional modal logic, different properties of the relation in the models yield different Hilbert systems. Early axiomatisation results [29,25,16] were tailored to the case where the betterness relation comes with many properties. These have been criticized as being too demanding in some contexts. Therefore subsequent research investigated how to extend these results to models equipped with a betterness relation meeting less conditions, if any at all [10,21]. Åqvist's system **E**, corresponds to the most general case, involving no commitment to any structural property of the relation. Stronger systems–like **F** and **G**–are obtained by adding extra constraints on the betterness relation. (A roadmap of existing systems is, e.g., in [10,22].) In this paper we focus on **E**, the weakest known preference-based dyadic deontic logic.

So far for preference-based deontic logics there has been an almost exclusive focus on the connection between semantic properties and Hilbert systems. Very little research has been done on Gentzen-style calculi. To our knowledge only **G**, due to its equivalence with Lewis's VTA and van Fraassen's CD, has an analytic Gentzen calculus [9]. As is well known such calculi have significant practical and theoretical advantages compared to Hilbert systems. In analytic calculi proof search proceeds indeed by step-wise decomposition of the formulas to be proven. For this reason they can be employed to establish important metalogical properties for the formalized logics (e.g., decidability, complexity and interpolation), and facilitate the development of automated reasoning methods. In general, analytic calculi serve to find derivations and hence provide forms of constructive *explanations* for normative systems; e.g. showing which hypotheses have been used in deriving certain obligations given specific facts. They also facilitate counter-model construction from non-derivable statements, and hence provide explanations of why "something should not be done".

The present paper aims at filling in this gap, focusing on Åqvist's system **E**. We introduce an analytic Gentzen-style calculus **HE** for **E**, and use (a reformulation of) it to provide an alternative decidability proof for **E** and a complexity result. The calculus is also employed to generate formal explanations for a well-known CTD paradox [7] from the deontic logic literature.

HE admits the elimination of the key rule of cut−which simulates Modus Ponens in Hilbert systems−and the consequent (relaxed version of the) subformula property; moreover its completeness proof is independent from the logic's semantics. An "optimized" version **HE**+ of **HE** is also given, that supports automated proof search and counterexample constructions.[3] **HE**+ is used to prove that the validity problem of **E** is co-NP and countermodels have polynomial size.

We highlight two salient features of our approach.

- Since **E** is tightly connected with the modal logic S5 ($S5$ is actually a sublogic of **E**), our calculi are defined using the hypersequent framework [4]−a simple extension of Gentzen' sequent framework−needed to provide a cut-free calculus for $S5$ [19,4,14], i.e. a calculus in which the cut rule is redundant.
- Similarly to previous work on modal interpretation of conditionals, *e.g.*, [8,27], we encode maximality by a unary modal operator. Intuitively the fact that x is among the best worlds that force a formula A may be understood as saying that all the worlds accessible from x via the betterness relation (or "above" x according to the ranking) force not-A. This is encoded as $\mathcal{B}et\neg A$, where $\mathcal{B}et$ is a K-type modal operator. The conditional obligation $\bigcirc(B/A)$ can be indirectly defined as $\Box(A \wedge \mathcal{B}et\neg A \to B)$, where $\Box$ obeys the laws of S5. Here "indirectly" indicates that the reduction schema is not explicitly introduced. $\mathcal{B}et$ is not part of the language of **E**, but is used at the meta-level in the Gentzen-style system to define suitable rules for the conditional.

We remark that although our calculus in some sense "translates" **E** into the bi-modal logic $S5+K$, its complexity turns out to be the same as for classical logic: co-NP; this contrasts with the P-SPACE complexity of $S5+K$.

2 System E

In this section we present the logic **E** both syntactically and semantically.

Definition 1. *The language $\mathcal{L}$ is defined by the following BNF:*

$$A ::= p \in PropVar \mid \neg A \mid A \to A \mid \Box A \mid \bigcirc(A/A)$$

$\Box A$ is read as "A is settled as true", and $\bigcirc(B/A)$ as "B is obligatory, given A". The Boolean connectives other than $\neg$ and $\to$ are defined as usual.

Definition 2. *The axiomatization of* **E** *consists of any Hilbert system for classical propositional logic, the Modus Ponens rule (MP): If $\vdash A$ and $\vdash A \to B$ then $\vdash B$, the rule (Nec): If $\vdash A$ then $\vdash \Box A$ and the following axioms:*

$$\textit{S5 axioms for } \Box \qquad \text{(S5)}$$
$$\bigcirc(B \to C/A) \to (\bigcirc(B/A) \to \bigcirc(C/A)) \qquad \text{(COK)}$$
$$\bigcirc(A/A) \qquad \text{(Id)}$$
$$\bigcirc(C/A \wedge B) \to \bigcirc(B \to C/A) \qquad \text{(Sh)}$$
$$\Box(A \leftrightarrow B) \to (\bigcirc(C/A) \leftrightarrow \bigcirc(C/B)) \qquad \text{(Ext)}$$
$$\bigcirc(B/A) \to \Box\bigcirc(B/A) \qquad \text{(Abs)}$$
$$\Box A \to \bigcirc(A/B) \qquad \text{(O-Nec)}$$

The notions of derivation and theoremhood are as usual.

[3] See [5] for an alternative method for generating countermodels.

An intuitive reading of the axioms is as follows. A basic design choice of the logic **E** is that necessity is interpreted as in the modal logic S5. (COK) is the conditional analogue of the familiar distribution axiom K. (Abs) is the absoluteness axiom of [16], and reflects the fact that the ranking is not world-relative. (O-Nec) is the deontic counterpart of the necessitation rule. (Ext) permits the replacement of necessarily equivalent sentences in the antecedent of deontic conditionals. (Id) is the deontic analogue of the identity principle. Named after Shoham [24, p. 77] who seems to have been the first to discuss it, (Sh) can be seen as expressing a "half" of deduction theorem or a "half" residuation property. The question of whether (Id) is a reasonable law for deontic conditionals has been much debated (see [23] for a defense).

The semantics of **E** can be defined in terms of *preference models*. They are possible-world models equipped with a comparative goodness relation $\succ$ on worlds so that $x \succ y$ can be read as "world x is *better* than world y". Conditional obligation is defined by considering "best" worlds: intuitively, $\bigcirc(B/A)$ holds in a model, if all the best worlds in which A is true also make B true.

Definition 3. *A preference model is a structure $M = (W, \succ, V)$ ($W \neq \emptyset$) whose members are called possible worlds, $\succ \subseteq W \times W$, $V : W \to \mathcal{P}(PropVar)$. The following evaluation rules are used, for all $x \in W$:*

- $M, x \vDash p$ *iff* $p \in V(x)$
- $M, x \vDash \neg A$ *iff* $M, x \nvDash A$
- $M, x \vDash A \to B$ *iff if* $M, x \vDash A$ *then* $M, x \vDash B$
- $M, x \vDash \Box A$ *iff* $\forall y \in W \ M, y \vDash A$
- $M, x \vDash \bigcirc(B/A)$ *iff* $\forall y \in \mathrm{best}(A) \ M, y \vDash B$

where $\mathrm{best}(A) = \{y \in W \mid M, y \models A$ *and there is no* $z \succ y$ *such that* $M, z \models A\}$. *A formula A is valid in a model M if for all worlds x in M, $M, x \models A$. A formula A is* valid *iff it is valid in every preference model.*

Observe that we do not assume any specific property of $\succ$.

To the purpose of the calculi developed in the following, we introduce the modality $\mathcal{B}et$, which will allow us to represent the "Best" worlds: $M, x \vDash \mathcal{B}et A$ iff $\forall y \succ x \ M, y \vDash A$. By this definition, we get $x \in \mathrm{best}(A)$ iff $M, x \models A$ and $M, x \models \mathcal{B}et \neg A$. However, the modality $\mathcal{B}et$ is not part of $\mathcal{L}$. As a notational convention, when no confusion arise, we write $x \vDash A$ for $M, x \vDash A$. The following result from [21] is needed for subsequent developments:

Theorem 1. **E** *is sound and complete w.r.t. the class of all preference models.*

The completeness proof in [21] uses another notion of maximality, call it best$'$, where $y \in \mathrm{best}'(A)$ iff $y \models A$ and $y \succ z$ for all z s. t. $z \models A$ and $z \succ y$. Although best and best$'$ are not equivalent, our result follows almost at once. Indeed, starting with a model $M = (W, \succ, V)$ in which obligations are evaluated using best$'$, one can derive an equivalent model $M' = (W, \succ', V)$ (with W and V the same) in which obligations are evaluated using best.[4]

[4] Put $x \succ' y$ iff $x \succ y$ and $y \not\succ x$. We can easily verify that an arbitrarily chosen world satisfies exactly the same formulas in both models, viz. for all worlds x, $M, x \models A$

We end this section with two remarks. The first one concerns reductions of conditional logics to modal logics. In the literature various such reductions have been introduced; perhaps the best-known is the embedding of conditional logic into S4 put forth by Lamarre and Boutilier (see the discussion in [18] and the references therein). There are similarities with their approach, but also important differences. They define indeed an embedding of a conditional logic, different from **E**, into S4. In contrast, we do *not* embed **E** into any (bi)modal logic. **E** contains an $S5$ modality as a primitive notion, whose meaning is independent from the dyadic modality $\bigcirc(B/A)$.

The second remark concerns the suitability of **E** to handle exceptions. Readers familiar with [23,13] may question this suitability. We think that **E** does provide a minimal account of exceptions. However, we agree with [28] that a more adequate treatment of exceptions within a preference-based framework calls for the combined use of a normality relation and a betterness relation.

3 A cut-free hypersequent calculus for E

We introduce the hypersequent calculus **HE** for the logic **E**. **HE** is defined in a modular way by adding to the calculus for the modal logic $S5$ suitable rules for the dyadic obligation, and the $\mathcal{B}et$ operator. Introduced in [19] to define a cut-free calculus for $S5$, hypersequents consist of sequents working in parallel.

Definition 4. *A* hypersequent *is a multiset* $\Gamma_1 \Rightarrow \Pi_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Pi_n$ *where, for all* $i = 1, \ldots, n$, $\Gamma_i \Rightarrow \Pi_i$ *is an ordinary sequent, called* component*.*

The hypersequent calculus **HE** is presented in Def. 5. It consists of initial hypersequents (i.e., axioms), logical/modal/deontic and structural rules. The latter are divided into *internal* and *external rules*. **HE** incorporates the sequent calculus for the modal logic S4 as a sub-calculus and adds an additional layer of information by considering a single sequent to live in the context of hypersequents. Hence all the axioms and rules of **HE** (but the external structural rules) are obtained by adding to each sequent a context G (or H), representing a possibly empty hypersequent. For instance, the (hypersequent version of the) axioms are $\Gamma, p \Rightarrow \Delta, p \,|\, G$. The external structural rules include ext. weakening (ew) and ext. contraction (ec) (see Fig. 1). These behave like weakening and contraction over whole hypersequent components. The hypersequent structure opens the possibility to define new such rules that allow the "exchange of information" between different sequents. It is this type of rules which increases the expressive power of hypersequent calculi compared to sequent calculi, allowing the definition of cut-free calculi for logics that seem to escape a cut-free sequent formulation (e.g., $S5$). An example of external structural rule is the ($s5$) rule in [14] (reformulated as ($s5'$) in Fig. 1 to account for the presence of $\bigcirc$), that allows the peculiar axiom of $S5$ to be derived as follows:

iff $M', x \models A$. (The sole purpose of this construction is to extend the result in [21] to the current setting.)

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\Box A \Rightarrow \Box\neg\Box A, \Box A}{\Box A \Rightarrow \ \mid\ \Rightarrow \Box\neg\Box A, \Box A}\ (\mathrm{s5}')}{\Rightarrow \neg\Box A \mid\ \Rightarrow \Box\neg\Box A, \Box A}\ (\neg\mathrm{R})}{\Rightarrow \Box\neg\Box A, \Box A \mid\ \Rightarrow \Box\neg\Box A, \Box A}\ (\Box\mathrm{R})}{\Rightarrow \Box\neg\Box A, \Box A}\ (\mathrm{ec})}{\Rightarrow \neg\Box A \rightarrow \Box\neg\Box A}\ (\rightarrow\mathrm{R})+(\neg\mathrm{L})$$

$$\frac{G}{G \mid \Gamma \Rightarrow \Pi}\ (ew) \qquad \frac{G \mid \Gamma \Rightarrow \Pi \mid \Gamma \Rightarrow \Pi}{G \mid \Gamma \Rightarrow \Pi}\ (ec) \qquad \frac{G \mid \Gamma^{\Box}, \Gamma^{O}, \Gamma' \Rightarrow \Pi'}{G \mid \Gamma \Rightarrow \ \mid \Gamma' \Rightarrow \Pi'}\ (s5')$$

Fig. 1. External structural rules

$$\frac{\Gamma^{\Box}, \Gamma^{O}, A, \mathcal{B}et\ \neg A \Rightarrow B \mid G}{\Gamma \Rightarrow \bigcirc(B/A), \Delta \mid G}\ (\bigcirc R) \qquad \frac{\Gamma^{\Box}, \Gamma^{O}, \Gamma^{b\downarrow} \Rightarrow A \mid G}{\Gamma \Rightarrow \Delta, \mathcal{B}et\ A \mid G}\ (\mathcal{B}et)$$

$$\frac{\Gamma^{\Box}, \Gamma^{O} \Rightarrow A \mid G}{\Gamma \Rightarrow \Delta, \Box A \mid G}\ (\Box R) \qquad \frac{\Gamma, \Box A, A \Rightarrow \Delta \mid G}{\Gamma, \Box A \Rightarrow \Delta \mid G}\ (\Box L)$$

$$\frac{\Gamma, \bigcirc(B/A) \Rightarrow \Delta, A \mid G \quad \Gamma, \bigcirc(B/A) \Rightarrow \Delta, \mathcal{B}et\ \neg A \mid G \quad \Gamma, \bigcirc(B/A), B \Rightarrow \Delta \mid G}{\Gamma, \bigcirc(B/A) \Rightarrow \Delta \mid G}\ (\bigcirc L)$$

Fig. 2. Deontic and modal rules

The rules in Fig. 1 and 2 make use of the following notation:
$\Sigma^{b\downarrow} = \{G : \mathcal{B}et\ G \in \Sigma\}$ $\quad \Sigma^{\Box} = \{\Box G :\ \Box G \in \Sigma\}$
$\Sigma^{O} = \{\bigcirc(C/D) : \bigcirc(C/D) \in \Sigma\}$

Definition 5. *The hypersequent calculus* **HE** *consists of the hypersequent version of Gentzen LK sequent calculus for propositional classical logic, the external structural rules in Fig. 1 and the modal and deontic rules in Fig. 2.*

A *derivation* in **HE** is a tree obtained by applying the rules bottom up. A *proof* $\mathcal{D}$ is a derivation whose leafs are axioms. This distinction will be used in Sect. 4.

The soundness of **HE** is proved with respect to preference models. Although we can interpret directly an hypersequent H into the semantics, it is easier (and more readable) to interpret it as a formula $I(H)$ of the extended language $\mathcal{L}{+}\mathcal{B}et$ and show the validity of this formula whenever H is provable.

Theorem 2. *If there is a proof in* **HE** *of* $H := \Gamma_1 \Rightarrow \Pi_1 \mid \ldots \mid \Gamma_n \Rightarrow \Pi_n$*, then* $I(H) := \Box(\bigwedge \Gamma_1 \rightarrow \bigvee \Pi_1) \vee \ldots \vee \Box(\bigwedge \Gamma_n \rightarrow \bigvee \Pi_n)$ *is valid.*

Proof. By induction on the proof of H. We show $(\bigcirc R)$, $(\mathcal{B}et)$ and *(s5')*.

$(\bigcirc R)$ Suppose the premise is valid but not the conclusion. Thus for some model M and world x, $x \not\models \Box(\bigwedge \Gamma \rightarrow \bigvee \Delta \vee \bigcirc(B/A)) \vee \Box G$. Thus (1) $x \not\models \Box(\bigwedge \Gamma \rightarrow$

$\bigvee \Delta \vee \bigcirc(B/A))$ and $x \not\models \Box G$. Since the premise is valid: $x \models \Box(\bigwedge \Gamma^{\Box} \wedge \bigwedge \Gamma^{O} \wedge A \wedge \mathcal{B}et\ \neg A \to B) \vee \Box G$ so that (2) $x \models \Box(\bigwedge \Gamma^{\Box} \wedge \bigwedge \Gamma^{O} \wedge A \wedge \mathcal{B}et\ \neg A \to B)$. From (1) there is y s.t. (3) $y \models \bigwedge \Gamma$, $y \not\models \bigvee \Delta$ and $y \not\models \bigcirc(B/A)$; for the latter there is some z such that $z \in best(A)$ and $z \not\models B$ [evaluation rule for $\bigcirc$]. So $z \models A$ and $z \models \mathcal{B}et\neg A$ [def of $\mathcal{B}et$]. From (3), $y \models \bigwedge \Gamma^{\Box} \wedge \bigwedge \Gamma^{O}$, whence also for z, as $\Gamma^{\Box}$ and Γ^{O} express global assumptions, holding in all worlds in the model. Thus $z \models \bigwedge \Gamma^{\Box} \wedge \bigwedge \Gamma^{O} \wedge A \wedge \mathcal{B}et\neg A$. By (2) $z \models B$, a contradiction.

($\mathcal{B}et$) Suppose that the premise is valid but not the conclusion. Thus for a model M and world x (ignoring the context G) $x \not\models \Box(\bigwedge \Gamma \to \bigvee \Delta \vee \mathcal{B}etA)$, but (∗) $x \models \Box(\bigwedge \Gamma^{\Box} \wedge \bigwedge \Gamma^{O} \wedge \bigwedge \Gamma^{b\downarrow} \to A)$ thus for some world y: (1) $y \models \bigwedge \Gamma$ (2) $y \not\models \mathcal{B}etA$. Observe that (3) $y \models \bigwedge \Gamma^{\Box} \wedge \bigwedge \Gamma^{O}$ and that (4) $y \models \mathcal{B}et\ C$ for all $\mathcal{B}et\ C \in \Gamma$. By (2) there is z with $z \succ y$ s.t. $z \not\models A$. Hence $z \models \bigwedge \Gamma^{\Box} \wedge \bigwedge \Gamma^{O}$. But by (4) we also get $z \models \Gamma^{b\downarrow}$. Therefore by (*) we get $z \models A$, a contradiction.

(s5') Suppose that the premise is valid but not the conclusion. Thus for some M and x, $x \not\models \Box\neg \bigwedge \Gamma \vee \Box(\bigwedge \Gamma' \to \bigvee \Pi')$, so that $x \not\models \Box\neg \bigwedge \Gamma$ and $x \not\models \Box(\bigwedge \Gamma' \to \bigvee \Pi')$. Therefore there are $y, z \in W$, such that $y \not\models \neg \bigwedge \Gamma$, meaning (1) $y \models \bigwedge \Gamma$ and $z \not\models \bigwedge \Gamma' \to \bigvee \Pi'$, which entails (2) $z \models \bigwedge \Gamma'$ and (3) $z \not\models \bigvee \Pi'$. By validity of the premise, $z \models \bigwedge \Gamma^{\Box} \wedge \bigwedge \Gamma^{O} \wedge \bigwedge \Gamma' \to \bigvee \Pi'$, so that by (3), (4) $z \not\models \bigwedge \Gamma^{\Box} \wedge \bigwedge \Gamma^{O} \wedge \bigwedge \Gamma'$. But by (1) , $z \models \bigwedge \Gamma^{\Box} \wedge \bigwedge \Gamma^{O}$ so that by (2) and (4) we have a contradiction.

Theorem 3 (Completeness with cut). *Each theorem of* **E** *has a proof in* **HE** *with the addition of the cut rule:*

$$\dfrac{G \mid \Gamma, A \Rightarrow \Delta \qquad H \mid \Sigma \Rightarrow \Pi, A}{G \mid H \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi}\ (cut)$$

Proof. As Modus Ponens corresponds to the provability of $A, A \to B \Rightarrow B$ and two applications of cut, it suffices to show that (Nec) and all the axioms of **E** are provable in **HE**. As an example, we show a proof of (COK):

$$\dfrac{\dfrac{\dfrac{\dfrac{B \to C, B \Rightarrow C \quad \dfrac{A \Rightarrow A}{\mathcal{B}et\ \neg A \Rightarrow \mathcal{B}et\ \neg A}(\mathcal{B}et)^{\star} \quad A \Rightarrow A}{B \to C, \bigcirc(B/A), A, \mathcal{B}et\ \neg A \Rightarrow C}(\bigcirc L)* \quad \dfrac{A \Rightarrow A}{\mathcal{B}et\neg A \Rightarrow \mathcal{B}et\ \neg A}(\mathcal{B}et)^{\star} \quad A \Rightarrow A}{\bigcirc(B \to C/A), \bigcirc(B/A), A, \mathcal{B}et\ \neg A \Rightarrow C}(\bigcirc L)*}{\bigcirc(B \to C/A), \bigcirc(B/A) \Rightarrow \bigcirc(C/A)}(\bigcirc R)}{\Rightarrow \bigcirc(B \to C/A) \to (\bigcirc(B/A) \to \bigcirc(C/A))}(\to R)\text{x2}$$

(∗ in the above proof stands for additional applications of internal weakening, and $(\mathcal{B}et)^{\star}$ stands for $(\mathcal{B}et) + (\neg L) + (\neg R)$)

Cut-elimination

Theorem 3 heavily relies on the presence of the cut rule. In this section we give a constructive proof that cut can in fact be *eliminated* from **HE** proofs. This result (cut elimination) implies (a relaxed form of) the *subformula property*: all formulas occurring in a cut-free **HE** proof are subformulas (possibly negated and under the scope of $\mathcal{B}et$) of the formulas to be proved.

Proof idea: To reduce the complexity of a cut on a formula of the form $\neg A$ or $A \to B$ we can exploit the rule invertibilities (Lemma 1). Some care is needed to deal with cut-formulas of the form $\Box A$, $\mathcal{B}et\ A$ and $\bigcirc(B/A)$. There we cannot use the invertibility argument and cuts have to be shifted upward till the cut-formula is introduced. Notice however that the $(\Box R)$, $(\bigcirc R)$ and $(\mathcal{B}et)$ rules do not allow to shift *every* cut upwards: only those involving sequents of a certain "good" shape. The proof hence proceeds by shifting uppermost cuts upwards in a specific order: first over the premise in which the cut formula appears on the right (Lemma 4) and then, when a rule introducing the cut formula is reached (and in this case the sequent has a "good" shape), shifting the cut upwards over the other premise (Lemma 3) till the left cut formula is introduced and the cut can be replaced by smaller cuts. The hypersequent structure does not require major changes; as the $(s5')$ rule allows cuts with "good" shaped sequents to be shifted upwards, to handle (ec) we consider the hypersequent version of the multicut: cutting one component (i.e. sequent) against possibly many components.

The *length* $|\mathcal{D}|$ of an **HE** proof $\mathcal{D}$ is (the maximal number of applications of inference rules) $+1$ occurring on any branch of d. The *complexity* $\ulcorner A \urcorner$ of a formula A is defined as: $\ulcorner A \urcorner = 0$ if A is atomic, $\ulcorner \neg A \urcorner = \ulcorner A \urcorner + 1$, $\ulcorner A \to B \urcorner = \ulcorner A \urcorner + \ulcorner B \urcorner + 1$, $\ulcorner \mathcal{B}et\ A \urcorner = \ulcorner A \urcorner + 1$, $\ulcorner \Box A \urcorner = \ulcorner A \urcorner + 1$, and $\ulcorner \bigcirc(A/B) \urcorner = \ulcorner A \urcorner + \ulcorner B \urcorner + 3$. The *cut rank* $\rho(\mathcal{D})$ of $\mathcal{D}$ is the maximal complexity $+\ 1$ of cut formulas in $\mathcal{D}$, noting that $\rho(\mathcal{D}) = 0$ if $\mathcal{D}$ is cut-free. We use A^n to indicate n occurrences of A.

It is easy to see that the rules of the classical propositional connectives remain invertible, as stated in the lemma below.

Lemma 1. *Given an* **HE** *proof* $\mathcal{D}$ *of a hypersequent containing a compound formula* $\neg A$ *(resp.* $A \to B$*), we can find a proof* $\mathcal{D}'$ *of the same hypersequent ending in an introduction rule for* $\neg A$ *(resp.* $A \to B$*) and with* $\rho(\mathcal{D}') \leq \rho(\mathcal{D})$*.*

In **HE** any cut whose cut formula is immediately introduced in left and right premise can be replaced by smaller cuts. More formally,

Lemma 2. *Let* A *be a compound formula and* $\mathcal{D}_l$ *and* $\mathcal{D}_r$ *be* **HE** *proofs such that* $\rho(\mathcal{D}_l) \leq \ulcorner A \urcorner$ *and* $\rho(\mathcal{D}_r) \leq \ulcorner A \urcorner$*, and*

1. $\mathcal{D}_l$ *is a proof of* $G \mid \Gamma, A \Rightarrow \Delta$ *ending in a rule introducing* A
2. $\mathcal{D}_r$ *is a proof of* $H \mid \Sigma \Rightarrow A, \Pi$ *ending in a rule introducing* A

We can find an **HE** *proof of* $G \mid H \mid \Gamma, \Sigma \Rightarrow \Delta, \Pi$ *with* $\rho(\mathcal{D}) \leq \ulcorner A \urcorner$*.*

Proof. We show the only non-trivial case: $A = \mathcal{B}et\ B$, where a cut

$$\dfrac{\dfrac{H \mid \Sigma^{\Box}, \Sigma^{O}, \Sigma^{b\downarrow}, B \Rightarrow C}{G \mid \Sigma, \mathcal{B}et\ B \Rightarrow \mathcal{B}et\ C, \Pi}\ (\mathcal{B}\text{et}) \qquad \dfrac{H \mid \Gamma^{\Box}, \Gamma^{O}, \Gamma^{b\downarrow} \Rightarrow B}{H \mid \Gamma \Rightarrow \mathcal{B}et\ B, \Delta}\ (\mathcal{B}\text{et})}{G \mid H \mid \Gamma, \Sigma \Rightarrow \mathcal{B}et\ C, \Delta, \Pi}\ (\text{cut})$$

is replaced by

$$\dfrac{\dfrac{H \mid \Sigma^{\Box}, \Sigma^{O}, \Sigma^{b\downarrow}, B \Rightarrow C \qquad G \mid \Gamma^{\Box}, \Gamma^{O}, \Gamma^{b\downarrow} \Rightarrow B}{G \mid H \mid \Sigma^{\Box}, \Sigma^{O}, \Sigma^{b\downarrow}\Gamma^{\Box}, \Gamma^{O}, \Gamma^{b\downarrow} \Rightarrow C}\,(\text{cut})}{G \mid H \mid \Gamma, \Sigma \Rightarrow \mathcal{B}et\ C, \Delta, \Pi}\,(\mathcal{B}\text{et})$$

Lemma 3. *Let $\mathcal{D}_l$ and $\mathcal{D}_r$ be* **HE** *proofs such that:*

1. *$\mathcal{D}_l$ is a proof of $G \mid \Gamma_1, A^{\lambda_1} \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n, A^{\lambda_n} \Rightarrow \Delta_n$;*
2. *A is a compound formula and $\mathcal{D}_r := H \mid \Sigma \Rightarrow A, \Pi$ ends with a right logical rule introducing an indicated occurrence of A*
3. *$\rho(\mathcal{D}_l) \leq \ulcorner A \urcorner$ and $\rho(\mathcal{D}_r) \leq \ulcorner A \urcorner$;*

Then we can construct an **HE** *proof $\mathcal{D}$ of $G \mid H \mid \Gamma_1, \Sigma^{\lambda_1} \Rightarrow \Delta_1, \Pi^{\lambda_1} \mid \ldots \mid \Gamma_n, \Sigma^{\lambda_n} \Rightarrow \Delta_n, \Pi^{\lambda_n}$ with $\rho(\mathcal{D}) \leq \ulcorner A \urcorner$.*

Proof. We distinguish cases according to the shape of A. If A is of the form $\neg B$ or $B \rightarrow C$, the claim follows by Lemmas 1 and 2. If A is $\Box B$, $\bigcirc(B/C)$ or $\mathcal{B}et\ B$ the proof proceeds by induction on $|\mathcal{D}_l|$. If $\mathcal{D}_l$ ends in an initial sequent, then we are done. If $\mathcal{D}_l$ ends in a left rule introducing one of the indicated cut formulas, the claim follows by (i.h. and) Lemma 2. Otherwise, let (r) be the last inference rule applied in $\mathcal{D}_l$. The claim follows by the i.h., an application of (r) and/or weakening. Some care is needed to handle the cases in which r is $(s5')$, $(\Box R)$, $(\bigcirc R)$ or $(\mathcal{B}et)$ and A is not in the hypersequent context G. Notice that when $A = \Box B$ (resp. $A = \bigcirc(B/C)$) the conclusion of $\mathcal{D}_r$ is $\Sigma \Rightarrow \Box B, \Pi$ (resp. $\Sigma \Rightarrow \bigcirc(B/C), \Delta$), but we can safely use the "good"-shaped sequent $\Sigma^{\Box}, \Sigma^{O} \Rightarrow \Box B$ (resp. $\Sigma^{\Box}, \Sigma^{O} \Rightarrow \bigcirc(B/C)$), that allows cuts to be shifted upwards over all **HE** rules, and we apply weakening afterwards. When $A = \mathcal{B}et\ B$, notice that the cut formula does not appear in the premises of these rules. For example let $(r) = (s5')$, $A = \Box B$, and $\mathcal{D}_l$ ends as follows

$$\dfrac{\begin{matrix}\vdots\ d_l' \\ G \mid \Gamma^{\Box}, \Box B, \Gamma^{O}, \Gamma' \Rightarrow \Pi' \mid \ldots \mid \Omega, \Box B \Rightarrow \Delta\end{matrix}}{G \mid \Gamma, \Box B \Rightarrow \mid \Gamma' \Rightarrow \Pi' \mid \ldots \mid \Omega, \Box B \Rightarrow \Delta}\,(s5')$$

The claim follows by i.h. applied to the conclusion $G \mid \Gamma^{\Box}, \Box B, \Gamma^{O}, \Gamma' \Rightarrow \Pi' \mid \ldots \mid \Omega, \Box B \Rightarrow \Delta$ of d_l' (and $\Sigma^{O}, \Sigma^{\Box} \Rightarrow \Box B$), followed by an application of $(s5')$ and weakening. The case $A = \bigcirc(B/C)$ is the same. The cases involving $(\Box R)$, $(\bigcirc R)$ and $(\mathcal{B}et)$ are handled in a similar way.

Lemma 4. *Let $\mathcal{D}_l$ and $\mathcal{D}_r$ be* **HE** *proofs such that:*

1. *$\mathcal{D}_l$ is a proof of $G \mid \Gamma, A \Rightarrow \Delta$;*
2. *$\mathcal{D}_r$ is a proof of $H \mid \Sigma_1 \Rightarrow A^{\lambda_1}, \Pi_1' \mid \ldots \mid \Sigma_n \Rightarrow A^{\lambda_n}, \Pi_n'$;*
3. *$\rho(\mathcal{D}_l) \leq \ulcorner A \urcorner$ and $\rho(\mathcal{D}_r) \leq \ulcorner A \urcorner$.*

Then a proof $\mathcal{D}$ can be constructed in **HE** *of $G \mid H \mid \Sigma_1, \Gamma^{\lambda_1} \Rightarrow \Pi_1', \Delta^{\lambda_1} \mid \ldots \mid \Sigma_n, \Gamma^{\lambda_n} \Rightarrow \Pi_n', \Delta^{\lambda_n}$ with $\rho(\mathcal{D}) \leq \ulcorner A \urcorner$.*

Proof. Let (r) be the last inference rule applied in $\mathcal{D}_r$. If (r) is an axiom, then the claim holds trivially. Otherwise, we proceed by induction on $|\mathcal{D}_r|$, using Lemma 3 when (one of) the indicated occurrence(s) of A is principal. Assume A is not principal. If (r) acts only on H or is a rule other than $(s5')$, $(\Box R)$, $(\bigcirc R)$ and $(\mathcal{B}et)$ the claim follows by the i.h. and an application of (r). For the remaining rules notice that A is not in the rule premise (in case of $(s5')$ the "critical" component in the conclusion has empty right-hand side), hence the claim follows by applying (the i.h. to the other components, and) the respective rule followed by weakening.

Theorem 4 (Cut Elimination). *Cut elimination holds for* **HE**.

Proof. Let $\mathcal{D}$ be an **HE** proof with $\rho(\mathcal{D}) > 0$. We proceed by a double induction on $\langle \rho(\mathcal{D}), n\rho(\mathcal{D})\rangle$, where $n\rho(\mathcal{D})$ is the number of applications of cut in $\mathcal{D}$ with cut rank $\rho(\mathcal{D})$. Consider an uppermost application of (cut) in $\mathcal{D}$ with cut rank $\rho(\mathcal{D})$. By applying Lemma 4 to its premises either $\rho(\mathcal{D})$ or $n\rho(\mathcal{D})$ decreases.

Corollary 1 (Completeness). *Each theorem of* **E** *has a proof in* **HE**.

4 A proof search oriented calculus for E

The properties of the calculus **HE** include modularity, cut-elimination and a completeness proof which is independent from the semantics of **E**. However **HE** supports neither automated proof search nor counterexample constructions.

Here we introduce the calculus $\mathbf{HE}^+$ having terminating proof search, thereby providing a decision procedure for **E**, and in case of termination with failure a countermodel of the starting formula can be extracted checking a *single* failed derivation. Similarly to the calculus for $S5$ in [15], $\mathbf{HE}^+$ is obtained by making in **HE** all rules invertible, and all structural rules (including the external ones) admissible. Looking at the rules bottom up, this is achieved by copying the introduced formulas and the component containing it in the rule premises; the "simulation" of $(s5')$ is obtained by introducing additional left rules for $\Box$ and $\bigcirc(A/B)$ which add subformulas to different components of the hypersequent.

Using $\mathbf{HE}^+$ we will show that the validity problem of **E** is co-NP.

Definition 6. *The* $\mathbf{HE}^+$ *calculus consists of: the initial hypersequents* $\Gamma, p \Rightarrow \Delta, p \,|\, G$*, together with the following rules:*

- *Rules for the propositional connectives that repeat the introduced formulas in the premises, for example*

$$\frac{\Gamma, A \to B \Rightarrow \Delta, A \,|\, G \quad \Gamma, A \to B, B \Rightarrow \Delta \,|\, G}{\Gamma, A \to B \Rightarrow \Delta \,|\, G}\ (\to L) \qquad \frac{\Gamma, A \Rightarrow \Delta, A \to B, B, \,|\, G}{\Gamma \Rightarrow \Delta, A \to B \,|\, G}\ (\to R)$$

- *Rules for O*

$$\frac{\Gamma \Rightarrow \bigcirc(B/A), \Delta \mid A, \mathcal{B}et\ \neg A \Rightarrow B \mid G}{\Gamma \Rightarrow \bigcirc(B/A), \Delta \mid G}\ (\bigcirc R+)$$

$$\frac{\Gamma, \bigcirc(B/A) \Rightarrow \Delta, A \mid G \quad \Gamma, \bigcirc(B/A) \Rightarrow \Delta, \mathcal{B}et\ \neg A \mid G \quad \Gamma, \bigcirc(B/A), B \Rightarrow \Delta \mid G}{\Gamma, \bigcirc(B/A) \Rightarrow \Delta \mid G}\ (\bigcirc L+)$$

$$\frac{\Gamma, \bigcirc(B/A) \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi, A \mid G \quad \Gamma, \bigcirc(B/A) \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi, \mathcal{B}et\ \neg A \mid G \quad \Gamma, \bigcirc(B/A) \Rightarrow \Delta \mid \Sigma, B \Rightarrow \Pi \mid G}{\Gamma, \bigcirc(B/A) \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi \mid G}\ (\bigcirc L2)$$

– *Rule for $\mathcal{B}et$*

$$\frac{\Gamma \Rightarrow \Delta, \mathcal{B}et\ A \mid \Gamma^{b\downarrow} \Rightarrow A \mid G}{\Gamma \Rightarrow \Delta, \mathcal{B}et\ A \mid G}\ (\mathcal{B}et+)$$

– *Rules for $\Box$*

$$\frac{\Gamma \Rightarrow \Delta, \Box A \mid\ \Rightarrow A \mid G}{\Gamma \Rightarrow \Delta, \Box A \mid G}\ (\Box R+) \quad \frac{\Gamma, \Box A, A \Rightarrow \Delta \mid G}{\Gamma, \Box A \Rightarrow \Delta \mid G}\ (\Box L+) \quad \frac{\Gamma, \Box A \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi \mid G}{\Gamma, \Box A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi \mid G}\ (\Box L2)$$

The notion of proof and derivation is as for **HE**. The following lemma collects standard structural properties of $\mathbf{HE}^+$.

Lemma 5. *(i) All rules of $\mathbf{HE}^+$ are height-preserving invertible. (ii) rules applications permute over each other (with the usual exceptions). (iii) Internal and external weakening and contraction are admissible in $\mathbf{HE}^+$.*

Proof. (i) Follows by the fact that the premises already contain the conclusion. (ii) and (iii) are standard (and hence omitted).

As a consequence of this lemma the order of application of the rules is irrelevant.

Theorem 5. *If there is a proof of H in $\mathbf{HE}^+$ then $I(H)$ is valid*

Proof. We first show that the rules of $\mathbf{HE}^+$ can be simulated in **HE**. This holds for all the $\mathbf{HE}^+$ rules but ($\bigcirc L2$) and ($\Box L2$) by simply applying weakening, internal and external contraction. For ($\Box L2$) we have

$$\dfrac{\dfrac{\dfrac{\dfrac{G \mid \Gamma, \Box A \Rightarrow \Delta \mid \Sigma, A \Rightarrow \Pi}{G \mid \Gamma, \Box A \Rightarrow \Delta \mid \Sigma, \Box A \Rightarrow \Pi}\ (\Box\mathrm{L})}{G \mid \Gamma, \Box A \Rightarrow \Delta \mid \Box A \Rightarrow \mid \Sigma \Rightarrow \Pi}\ (\mathrm{S5}')}{G \mid \Gamma, \Box A \Rightarrow \Delta \mid \Gamma, \Box A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}\ (\mathrm{w})}{G \mid \Gamma, \Box A \Rightarrow \Delta \mid \Sigma \Rightarrow \Pi}\ (\mathrm{ec})$$

The argument for ($\bigcirc L2$) is analogous. The claim follows by Theorem 2.

We have adopted a "kleene'd" formulation of the calculus to make easier countermodel construction and termination of proof-search. They are both based on the notion of *saturation* that we define next. Given a hypersequent H, we write $\Gamma \Rightarrow \Delta \in H$ to indicate that $\Gamma \Rightarrow \Delta$ is a compontent of H.

Definition 7 (Saturation). *A hypersequent H is* saturated *if it is not an axiom and satisfies the following conditions associated to each rule application*

$(\to L)_S$ *if* $\Gamma, A \to B \Rightarrow \Delta \in H$ *then either* $A \in \Delta$ *or* $B \in \Gamma$
$(\to R)_S$ *if* $\Gamma \Rightarrow \Delta, A \to B \in H$ *then* $A \in \Gamma$ *and* $B \in \Delta$
$(\neg L)_S$ *if* $\Gamma, \neg A \Rightarrow \Delta \in H$ *then* $A \in \Delta$
$(\neg R)_S$ *if* $\Gamma \Rightarrow \Delta, \neg A \in H$ *then* $A \in \Gamma$
$(\bigcirc L+)_S$ *if* $\Gamma, \bigcirc(B/A) \Rightarrow \Delta \in H$ *then either* $A \in \Delta$ *or* $\mathcal{B}et\neg A \in \Delta$ *or* $B \in \Gamma$
$(\bigcirc L2)_S$ *if* $\Gamma, \bigcirc(B/A) \Rightarrow \Delta \in H$ *and* $\Sigma \Rightarrow \Pi \in H$ *then either* $A \in \Pi$ *or* $\mathcal{B}et\neg A \in \Pi$ *or* $B \in \Sigma$
$(\bigcirc R+)_S$ *if* $\Gamma \Rightarrow \bigcirc(B/A), \Delta \in H$ *then there is* $\Sigma \Rightarrow \Pi \in H$ *such that* $A \in \Sigma$, $\mathcal{B}et\neg A \in \Sigma$, *and* $B \in \Pi$
$(\mathcal{B}et+)_S$ *if* $\Gamma \Rightarrow \Delta, \mathcal{B}et A \in H$ *then there is* $\Sigma \Rightarrow \Pi \in H$ *such that* $\Gamma^{b\downarrow} \subseteq \Sigma$ *and* $A \in \Pi$
$(\Box R+)_S$ *if* $\Gamma \Rightarrow \Delta, \Box A \in H$ *then there is* $\Sigma \Rightarrow \Pi \in H$ *such that* $A \in \Pi$
$(\Box L+)_S$ *if* $\Gamma, \Box A \Rightarrow \Delta \in H$ *then* $A \in \Gamma$
$(\Box L2)_S$ *if* $\Gamma, \Box A \Rightarrow \Delta \in H$ *and* $\Sigma \Rightarrow \Pi \in H$ *then* $A \in \Sigma$

The key to obtain termination is to avoid the application of a rule to hypersequents which in a sense already contain the premise of that rule.

Definition 8 (Redundant application). *A backward application of a rule (R) to an hypersequent H is* redundant *if H satisfies the saturation condition $(R)_S$ associated to that application of (R).*

We call a derivation/proof *irredundant* if (i) no rule is applied to an axiom, and (ii) it does not contain any redundant application of rule. It is easy to see that by the admissibility of internal weakening and external contraction (Lemma 5) redundant applications of the rules can be safely removed.

Lemma 6. *Every hypersequent provable in* $\mathbf{HE}^+$ *has an irredundant proof.*

Proof. By induction on the height of a uppermost redundant application. To illustrate the argument consider a redundant application of the (Bet+) rule

$$\frac{\Gamma \Rightarrow \Delta, \mathcal{B}et\ A \mid \Gamma^{b\downarrow} \Rightarrow A \mid \Gamma^{b\downarrow}, \Sigma' \Rightarrow \Pi', A \mid G}{\Gamma \Rightarrow \Delta, \mathcal{B}et\ A \mid \Gamma^{b\downarrow}, \Sigma' \Rightarrow \Pi', A \mid G}\ (\mathcal{B}et+)$$

this is transformed as follows:

$$\dfrac{\dfrac{\Gamma \Rightarrow \Delta, \mathcal{B}et\ A \mid \Gamma^{b\downarrow} \Rightarrow A \mid \Gamma^{b\downarrow}, \Sigma' \Rightarrow \Pi', A \mid G}{\Gamma \Rightarrow \Delta, \mathcal{B}et\ A \mid \Gamma^{b\downarrow}, \Sigma' \Rightarrow \Pi', A \mid \Gamma^{b\downarrow}, \Sigma' \Rightarrow \Pi', A \mid G}\ (Wk)}{\Gamma \Rightarrow \Delta, \mathcal{B}et\ A \mid \Gamma^{b\downarrow}, \Sigma' \Rightarrow \Pi', A \mid G}\ (ec)$$

The above property justifies the restriction to irredundant proofs from a syntactical point of view, although this justification is not really needed for completeness (Theorem 7 below).

We now use the calculus $\mathbf{HE}^+$ to give a decision procedure for the logic $\mathbf{E}$; the key issue here is to restrict proof-search to irredundant derivations.

We denote by $|A|$ the size of a formula A considered as a string of symbols.

Theorem 6. *Every $\mathbf{HE}^+$ derivation of a formula A of $\mathbf{E}$ is finite and it is either a proof or it contains a saturated hypersequent.*

Proof. Let $\mathcal{D}$ be any derivation built from $\Rightarrow A$ by backwards application of the rules. We first prove that all hypersequents in $\mathcal{D}$ are finite and provide an upper bound on their size. To this purpose let $|A| = n$ and consider $SUB^+(A) = \{B \mid B \text{ is a subformula of } A\ \} \cup \{\mathcal{B}et\neg C \mid \bigcirc(D/C) \text{ occurs in } A, \text{for some } C\}$. Clearly the cardinality of $SUB^+(A)$ is $O(n)$ and so it is the size of each formula in it.

Let $H := \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_k \Rightarrow \Delta_k$ be any hypersequent occurring in $\mathcal{D}$. The size of each component is bounded by $O(n^2)$: it contains $O(n)$ formulas each one of size $O(n)$. To estimate the size of H, we estimate the number of its components (i.e. k). Observe that the rules which "create" new components are $(\Box R+)$, $(\bigcirc R+)$, and $(\mathcal{B}et+)$. Consider first $(\Box R+)$: by the irredundancy restriction this rule is applied *exactly once* to each formula, say $\Box C$, occurring in the consequent of a component and creates only *one* new component, no matter if $\Box C$ appears in the consequent of many components. To illustrate the situation, consider, e.g.,

$$\frac{\ldots \Gamma_i \Rightarrow \Delta_i, \Box C \mid \ \Rightarrow C \mid \ldots \mid \Gamma_j \Rightarrow \Delta_j, \Box C \mid \ldots \mid \Gamma_k \Rightarrow \Delta_k}{\ldots \Gamma_i \Rightarrow \Delta_i, \Box C \mid \ldots \mid \Gamma_j \Rightarrow \Delta_j, \Box C \mid \ldots \mid \Gamma_k \Rightarrow \Delta_k}$$

the irredundancy restriction ensures that if $(\Box R+)$ is applied to $\Gamma_i \Rightarrow \Delta_i$, it cannot be applied to the component $\Gamma_j \Rightarrow \Delta_j, \Box C$. This means that the number of components created by $(\Box R+)$ is bounded by $\Box$-ed subformulas of A, whence it is $O(n)$. The situation for $(\bigcirc R+)$ is similar.

For the rule (Bet+), first observe the following fact:
Given any derivation $\mathcal{D}$ having at its root a formula of $\mathbf{E}$ (that is a hypersequent $\Rightarrow A$) at most one Bet formula can occur in the antecedent Γ_i of any component of any hypersequent in $\mathcal{D}$, that is $\Gamma_i^{b\downarrow}$ contains at most one formula.

By this fact the rule $(\mathcal{B}et+)$ may be applied when $\Gamma_i^{b\downarrow}$ contains a formula and when $\Gamma_i^{b\downarrow} = \emptyset$, in both cases the applications are not duplicated, for instance in the former case, we may have:

$$\frac{\ldots \Gamma_i, \mathcal{B}et\ \neg E \Rightarrow \mathcal{B}et\ \neg F \mid \neg E \Rightarrow \neg F \mid \ldots \mid \Gamma_j, \mathcal{B}et\ \neg E \Rightarrow \mathcal{B}et\ \neg F \mid \ldots}{\ldots \Gamma_i, \mathcal{B}et\ \neg E \Rightarrow \mathcal{B}et\ \neg F \mid \ldots \mid \Gamma_j, \mathcal{B}et\ \neg E \Rightarrow \mathcal{B}et\ \neg F \mid \ldots}$$

Thus there is at most one application of the $(\mathcal{B}et+)$ rule for any pair of $\mathcal{B}et$ formulas (case $\Gamma_i^{b\downarrow} \neq \emptyset$) plus possibly an application for any $\mathcal{B}et$ formula (case $\Gamma_i^{b\downarrow} = \emptyset$). Since $\mathcal{B}et$ formulas come from the decomposition of O-subformulas and there are $O(n)$ of them, the number of components created by the $(\mathcal{B}et+)$ rule is $O(n^2+n)=O(n^2)$. We can conclude that the number of components of any hypersequent in $\mathcal{D}$ is $O(n^2)$, whence the size of each hypersequent is $O(n^4)$.

We get also an upper bound on proof branches: since any backward application of a rule is irredundant, it must add some formula/component. Therefore the length of each proof branch is also bounded by $O(n^4)$ and the derivation is finite. Finally each leaf must be an axiom or a saturated hypersequent otherwise a rule would have been applied to it.

The next theorem shows the completeness of $\mathbf{HE}^+$.

Theorem 7. *Every valid formula A of* $\mathbf{E}$ *has a proof in* $\mathbf{HE}^+$.

Proof. We prove the contrapositive: if A is not provable in $\mathbf{HE}^+$ then there is a model in which A is not valid. Suppose that A is not provable in $\mathbf{HE}^+$, by the previous theorem any derivation of $\Rightarrow A$ as root contains at least one branch ending with a saturated hypersequent. Fix a derivation and let $H = \Gamma_1 \Rightarrow \Delta_1 \,|\, \ldots \,|\, \Gamma_n \Rightarrow \Delta_n$ be the intended saturated hypersequent. We build a countermodel of A based on H. First we enumerate the components of H, calling H' the corresponding structure:

$$H' = 1 : \Gamma_1 \Rightarrow \Delta_1 \,|\, 2 : \Gamma_2 \Rightarrow \Delta_2 \ldots \,|\, n : \Gamma_n \Rightarrow \Delta_n$$

We then define a model $M = (W, \succ, V)$ by stipulating:

$W = \{1, \ldots, n\}$, $\quad V(i) = \{P \mid P \in \Gamma_i\}$ with $i : \Gamma_i \Rightarrow \Delta_i \in H'$
$j \succ i$ where $i : \Gamma_i \Rightarrow \Delta_i \in H'$, $j : \Gamma_j \Rightarrow \Delta_j \in H'$, we have $\Gamma_i^{b\downarrow} \subseteq \Gamma_j$ and there is a formula $\mathcal{B}et\ C \in \Delta_i$ such that $C \in \Delta_j$.

Notice that in the definition of the preference relation it may be $i = j$. We now prove the fundamental claim (truth lemma); to this purpose we do not need to consider formulas with $\mathcal{B}et$, thus for $B \in \mathcal{L}$:

(a) for any $i \in W$, if $B \in \Gamma_i$ then $M, i \models B$
(b) for any $i \in W$, if $B \in \Delta_i$ then $M, i \not\models B$

Both claims (a) and (b) are proved by structural induction on B.

- Let B be an atom P, then (a) holds by definition of $V(i)$. Concerning (b), let $P \in \Delta_i$, since H is saturated, $P \notin \Gamma_i$, otherwise H would be an axiom; thus $P \notin V(i)$ whence $M, i \not\models P$.
- the propositional case use saturation conditions and induction hypothesis.
- Let $B = \bigcirc(D/C)$. (a) suppose $\bigcirc(D/C) \in \Gamma_i$. We have to show that for every $j \in W$ the following holds: (case 1) $M, j \not\models C$, or (case 2) there is $k \in W$ with $k \succ j$ such that $M, k \models C$, or (case 3) $M, j \models D$. By saturation conditions $(\bigcirc L+)_S$ or $(\bigcirc L2)_S$ according to $i = j$ or $i \neq j$, we have that either $C \in \Delta_j$ or $\mathcal{B}et\neg C \in \Delta_j$ or $D \in \Gamma_j$, in the first case by i.h. we get $M, j \not\models C$ (case 1), in the third case, by i.h. we get $M, j \models D$ (case 3). Thus we are left with the case $\mathcal{B}et\neg C \in \Delta_j$. By saturation condition $(\mathcal{B}et)_S$,

there is $k : \Gamma_k \Rightarrow \Delta_k \in H'$ such that $\Gamma_j^{b\downarrow} \subseteq \Gamma_k$ and $\neg C \in \Delta_k$. Observe that by construction it holds $k \succ j$. Moreover, by saturation condition $(\neg R)_S$, $C \in \Gamma_k$, whence by inductive hypothesis $M, k \models C$.
(b) Suppose $\bigcirc(D/C) \in \Delta_i$. We have to show that there is $j \in W$ such that: $M, j \models C$; for all $k \in W$ with $k \succ j$ $M, k \not\models C$; and $M, j \not\models D$. By $(OR)_S$ there is $j : \Gamma_j \Rightarrow \Delta_j \in H'$ such that $C \in \Gamma_j$, $\mathcal{B}et\neg C \in \Gamma_j$, and $D \in \Delta_j$; by i.h. we get $M, j \models C$ and $M, j \not\models D$. We have still to prove that for all $k \in W$ with $k \succ j$ $M, k \not\models C$. To this aim, suppose $k \succ j$, by construction we have that there is $j : \Gamma_k \Rightarrow \Delta_k \in H'$ such that $\Gamma_j^{b\downarrow} \subseteq \Gamma_k$ and for some formula $\mathcal{B}et\ E \in \Delta_j$ it holds $E \in \Delta_k$. Since $\mathcal{B}et\neg C \in \Gamma_j$, $\neg C \in \Gamma_j^{b\downarrow} \subseteq \Gamma_k$, whence by $(\neg L)_S$ $C \in \Delta_k$; by i.h. we conclude $M, k \not\models C$ and we are done.

- $B = \Box C$. (a) suppose $\Box C \in \Gamma_i$. We have to show that for every $j \in W$, $M, j \models C$. Let $j \in W$ this means that $j : \Gamma_k \Rightarrow \Delta_k \in H'$ (it might be $j = i$), by saturation condition $(\Box L+)_S$ or $(\Box L2)_S$, according to $i = j$ or $i \neq j$ we have $C \in \Gamma_j$, whence by i.h. $M, j \models C$.
(b) Suppose $\Box C \in \Delta_i$. By saturation condition $(\Box R+)_S$ there is $j : \Gamma_j \Rightarrow \Delta_j \in H'$ such that $C \in \Delta_j$, thus by i.h. $M, j \not\models C$.

Being $\Rightarrow A$ the root of the derivation, for some $i : \Gamma_i \Rightarrow \Delta_i \in H'$, we have $A \in \Delta_i$, and by claim (b) $M, i \not\models A$, showing that A is not valid in M.

This allows us to obtain a complexity bound for validity in **E**.

Theorem 8. *Validity of formula of* **E** *can be decided in Co-NP time.*

Proof. Given A, to decide whether A is valid, we consider a non-deterministic algorithm which takes as input $\Rightarrow A$ and guesses a saturated hypersequent H: if it finds it, the algorithm answers "non-valid", otherwise, it answers "valid". As shown in the proof of the Theorem 6, the size of the candidate saturated hypersequent H is polynomially bounded by the size of A ($= O(|A|^4)$), moreover checking whether H is saturated can also be done in polynomial time in the size of A (namely $O(|A|^8)$). More concretely, the algorithm can try to build H by applying the rules backwards in an arbirary but fixed order, applying the first applicable (i.e. non-redundant) rule and then choosing non-deterministically one of its premises if there are more than one. The number of steps is polynomially bounded by $O(|A|^4)$ and checking whether a rule is applicable to a given hypersequent is linear in the size of the hypersequent. Thus the whole non-deterministic computation is polynomial in the size of the input formula.

By the previous results **E** turns out to have the polysize model property.

Corollary 2. *If a formula A of* **E** *is satisfiable (that is $\neg A$ is not valid), then it has a model of polynomial size in the length of A.*

We end the section with an example of explanation, obtained by countermodel construction, of a well-known CTD paradox.

"Gentle Murder" [7]. Consider the following norms and fact: (i) You ought not kill (ii) If you kill, you ought to kill gently (iii) Killing gently is killing (iv) You kill. In many deontic logics, these sentences are inconsistent and in particular (ii)-(iv) allow to derive the obligation to kill, contradicting (i)–hence the "paradox". We formally show that this does not happen in the logic **E**. To this purpose let the above sentences be encoded by: $\bigcirc(\neg k/\top), \bigcirc(g/k), \Box(g \to k), k$ with the obvious meaning of propositional atoms. We first verify that the above formulas are consistent, thus we begin a derivation with root hypersequent

$$\bigcirc(\neg k/\top), \bigcirc(g/k), \Box(g \to k), k \Rightarrow \bot$$

One of the saturated hypersequents we find by applying the rules backwards is

$$\bigcirc(\neg k/\top), \bigcirc(g/k), \Box(g \to k), k, g \to k, g \Rightarrow \bot, \mathcal{B}et\neg\top \mid g \to k, \neg k \Rightarrow \neg\top, g$$

Following the construction of Theorem 7, we enumerate the components (respectively) by 1,2 and get the model $M = (W, \succ, V)$ where $W = \{1, 2\}$, the preference relation is $2 \succ 1$ and $V(1) = \{k, g\}, V(2) = \emptyset$. It is easy to see that $i \models g \to k$, for $i = 1, 2$, both $\bigcirc(\neg k/\top), \bigcirc(g/k)$ are valid in the model and $1 \models k$. Notice in particular that 1 is the "best" world where "kill" holds and in that world also "killing gently" holds.

We can also verify that the sentences (ii)-(iv) do not derive the obligation to kill. Notice that this claim in **E** is not entailed by what we have just proved. To this purpose we initialise the derivation by $\bigcirc(g/k), \Box(g \to k), k \Rightarrow \bigcirc(k/\top)$ and we get (among others) the following saturated hypersequent:

$$\bigcirc(g/k), \Box(g \to k), k, g \to k, g \Rightarrow \bigcirc(k/\top) \mid \top, \mathcal{B}et\neg\top, g \to k \Rightarrow k, g$$

We get the model $M = (W, \succ, V)$, where W and V are as before (1 and 2 are now constructed using the new hypersequent), but $\succ$ is empty meaning that all worlds are best. Now 2 is a "best" world in an absolute sense (i.e., for $\top$) and k does not hold there. By the evaluation rule (cf. Def. 3), $\bigcirc(k/\top)$ fails both in 1 and 2. Hence killing is not best overall, and you are not obliged to kill.

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References

1. C. Alchourrón. Philosophical foundations of deontic logic and the logic of defeasible conditionals. In J.-J. Meyer and R. Wieringa, editors, *Deontic Logic in Computer Science*, pages 43–84. John Wiley & Sons, Inc., New York, 1993.
2. L. Åqvist. Deontic logic. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic: Volume II*, pages 605–714. Springer, Dordrecht, 1984.
3. N. Asher and D. Bonevac. Common sense obligation. In Nute [20], pages 159–203.
4. A. Avron. The method of hypersequents in the proof theory of propositional non-classical logics. In *Logic: from foundations to applications*, pages 1–32. OUP, New York, 1996.

5. C. Benzmüller, A. Farjami, and X. Parent. Åqvist's dyadic deontic logic E in HOL. *IfCoLog*, 6:715–732, 2019.
6. S. Danielsson. *Preference and Obligation*. Filosofiska Färeningen, Uppsala, 1968.
7. J. Forrester. Gentle murder, or the adverbial samaritan. *J. of Phil.*, 81:193–197, 1984.
8. L. Giordano, V. Gliozzi, N. Olivetti, and G. L. Pozzato. Analytic tableaux calculi for KLM logics of nonmonotonic reasoning. *ACM Trans. Comput. Log.*, 10(3):18:1–18:47, 2009.
9. M. Girlando, B. Lellmann, N. Olivetti, and G. L. Pozzato. Standard sequent calculi for Lewis' logics of counterfactuals. In *Proc. JELIA*, pages 272–287, 2016.
10. L. Goble. Axioms for Hansson's dyadic deontic logics. *Filosofiska Notiser*, 6(1):13–61, 2019.
11. B. Hansson. An analysis of some deontic logics. *Noûs*, 3(4):373–398, 1969. Reprinted in [12, pp. 121-147].
12. R. Hilpinen, editor. *Deontic Logic*. Reidel, Dordrecht, 1971.
13. J. Horty. Deontic modals: Why abandon the classical semantics? *Pacific Philosophical Quarterly*, 95(4):424–460, 2014.
14. H. Kurokawa. Hypersequent calculi for modal logics extending S4. In *New Frontiers in Artificial Intelligence*, volume 8417 of *LNCS*, pages 51–68. Springer, 2013.
15. R. Kuznets and B. Lellmann. Grafting hypersequents onto nested sequents. *Log. J. IGPL*, 24(3):375–423, 2016.
16. D. Lewis. *Counterfactuals*. Blackwell, Oxford, 1973.
17. B. Loewer and M. Belzer. Dyadic deontic detachment. *Synthese*, 54:295–318, 1983.
18. D. Makinson. Five faces of minimality. *Studia Logica*, 52(3):339–379, 1993.
19. G. Minc. Some calculi of modal logic. *Trudy Mat. Inst. Steklov*, 98:88–111, 1968.
20. D. Nute, editor. *Defeasible Deontic Logic*. Kluwer, Dordrecht, 1997.
21. X. Parent. Completeness of Åqvist's systems E and F. *Rev. Symb. Log.*, 8(1):164–177, 2015.
22. X. Parent. Preference semantics for Hansson-type dyadic deontic logic: a survey of results. In *Handbook of Deontic Logic and Normative Systems*, volume 2, pages 7–70. College Publications, London, 2021.
23. H. Prakken and M. Sergot. Dyadic deontic logic and contrary-to-duty obligations. In Nute [20], pages 223–262.
24. Y. Shoham. *Reasoning About Change*. MIT Press, Cambridge, MA, USA, 1988.
25. W. Spohn. An analysis of Hansson's dyadic deontic logic. *J. of Phil. Logic*, 4(2):237–252, 1975.
26. J. Tomberlin. Contrary-to-duty imperatives and conditional obligation. *Noûs*, pages 357–375, 1981.
27. J. van Benthem, P. Girard, and O. Roy. Everything else being equal: A modal logic for ceteris paribus preferences. *J. of Phil. Logic*, 38(1):83–125, 2009.
28. L. van der Torre and Y.-H. Tan. The many faces of defeasibility in defeasible deontic logic. In Nute [20], pages 79–121.
29. B. van Fraassen. The logic of conditional obligation. *J. of Phil. Logic*, 1(3/4):417–438, 1972.