Appendix for: Cut-free Calculi and Relational Semantics for Temporal STIT logics

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Abstract. This paper is an appendix to the paper "Cut-free Calculi and Relational Semantics for Temporal STIT logics" by Berkel and Lyon, 2019 [2]. It provides the completeness proof for the basic STIT logic Ldm (relative to irreflexive, temporal Kripke STIT frames) as well as gives the derivation of the independence of agents axiom for the logic Xstit.

A Completeness of Ldm

We give the definitions and lemmas sufficient to prove the completeness of Ldm relative to Tstit frames [4,2]. We make use of the canonical model of Ldm (obtained by standard means [3,1]) to construct a Tstit model. A truth-lemma is then given relative to this model, from which, completeness follows as a corollary.

Definition 1 (Ldm-CS, Ldm-MCS). *A set $\Theta \subset \mathcal{L}_{\mathsf{Ldm}}$ is a* Ldm consistent set (Ldm-CS) *iff $\Theta \not\vdash_{\mathsf{Ldm}} \bot$. We call a set $\Theta \subset \mathcal{L}_{\mathsf{Ldm}}$ a* Ldm maximally consistent set (Ldm-MCS) *iff Θ is a* Ldm*-CS and for any set Θ' such that $\Theta \subset \Theta'$, $\Theta' \vdash_{\mathsf{Ldm}} \bot$.*

Lemma 1 (Lindenbaum's Lemma [3]). *Every* Ldm*-CS can be extended to a* Ldm*-MCS.*

Definition 2 (Present and Future Pre-Canonical Tstit Model). *The* present pre-canonical Tstit model *is the tuple $M^{\mathsf{pres}} = (W^{\mathsf{pres}}, \mathcal{R}_{\Box}^{\mathsf{pres}}, \{\mathcal{R}_i^{\mathsf{pres}} | i \in Ag\}, V^{\mathsf{pres}})$ defined below left, and the* future pre-canonical Tstit model *is the tuple $M^{\mathsf{fut}} = (W^{\mathsf{fut}}, \mathcal{R}_{\Box}^{\mathsf{fut}}, \{\mathcal{R}_i^{\mathsf{fut}} | i \in Ag\}, V^{\mathsf{fut}})$ defined below right:*

– W^{pres} *is the set of all* Ldm*-MCSs;*	– $W^{\mathsf{fut}} = W^{\mathsf{pres}}$*;*
– $\mathcal{R}_{\Box}^{\mathsf{pres}} wu$ *iff for all $\Box\phi \in w$, $\phi \in u$;*	– $\mathcal{R}_{\Box}^{\mathsf{fut}}(w) = \bigcap_{i \in Ag} \mathcal{R}_i^{\mathsf{pres}}(w)$*;*
– $\mathcal{R}_i^{\mathsf{pres}} wu$ *iff for all $[i]\phi \in w$, $\phi \in u$;*	– $\mathcal{R}_i^{\mathsf{fut}}(w) = \bigcap_{i \in Ag} \mathcal{R}_i^{\mathsf{pres}}(w)$*;*
– $V^{\mathsf{pres}}(p) = \{w \in W \mid p \in w\}$.	– $V^{\mathsf{fut}}(p) = V^{\mathsf{pres}}(p)$.

Definition 3 (Canonical Temporal Kripke STIT Model). *We define the* canonical temporal Kripke STIT model *to be the tuple $M^{\mathsf{Ldm}} = (W^{\mathsf{Ldm}}, \mathcal{R}_{\Box}^{\mathsf{Ldm}}, \{R_i^{\mathsf{Ldm}} | i \in Ag\}, \mathcal{R}_{Ag}^{\mathsf{Ldm}}, \mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}}, \mathcal{R}_{\mathsf{H}}^{\mathsf{Ldm}}, V^{\mathsf{Ldm}})$ such that:*

- $W^{\mathsf{Ldm}} = W^{\mathsf{pres}} \times \mathbb{N}$[1];
- $\mathcal{R}_{\Box}^{\mathsf{Ldm}} w^j u^j$ *iff (i)* $\mathcal{R}_{\Box}^{\mathsf{pres}} wu$ *and* $j = 0$, *or (ii)* $\mathcal{R}_{\Box}^{\mathsf{fut}} wu$ *and* $j > 0$;
- $\mathcal{R}_{i}^{\mathsf{Ldm}} w^j u^j$ *iff (i)* $\mathcal{R}_{i}^{\mathsf{pres}} wu$ *and* $j = 0$, *or (ii)* $\mathcal{R}_{i}^{\mathsf{fut}} wu$ *and* $j > 0$;
- $\mathcal{R}_{Ag}^{\mathsf{Ldm}}(w^j) = \bigcap_{1 \leq i \leq n} \mathcal{R}_{i}^{\mathsf{Ldm}}(w^j)$;
- $\mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}} = \{(w^j, w^k) | w^j, w^k \in W^{\mathsf{Ldm}} \text{ and } j < k\}$;
- $\mathcal{R}_{\mathsf{H}}^{\mathsf{Ldm}} = \{(u^i, w^i) | (w^i, u^i) \in \mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}}\}$;
- $V^{\mathsf{Ldm}}(p) = \{w^j \in W^{\mathsf{Ldm}} | w \in V^{\mathsf{pres}}(p)\}$.

Lemma 2. *For all* $\alpha \in \{\Box, Ag\} \cup Ag$, *if* $\mathcal{R}_{\alpha}^{\mathsf{Ldm}} w^j u^k$ *for* $j, k \in \mathbb{N}$, *then* $j = k$.

Proof. Follows by definition of the canonical Tstit model.

Lemma 3. *For all* $j \in \mathbb{N}$ *with* $k \geq 1$, $(w^j, u^j) \in \mathcal{R}_{Ag}^{\mathsf{Ldm}}$ *iff* $(w^{j+k}, u^{j+k}) \in \mathcal{R}_{Ag}^{\mathsf{Ldm}}$.

Proof. This follows from the fact that $u^0 \in \mathcal{R}_{Ag}^{\mathsf{Ldm}}(w^0)$ iff $u \in \bigcap_{i \in Ag} \mathcal{R}_{i}^{\mathsf{pres}}(w)$ iff $u \in \mathcal{R}_{i}^{\mathsf{fut}}(w)$ for each $i \in Ag$ iff $u \in \bigcap_{i \in Ag} \mathcal{R}_{i}^{\mathsf{fut}}(w)$ iff $u^k \in \bigcap_{i \in Ag} \mathcal{R}_{i}^{\mathsf{Ldm}}(w^k)$ for any $k > 0$.

Lemma 4 ([3]). *(i) For all* $\mathsf{x} \in \{\mathsf{pres}, \mathsf{fut}, \mathsf{Ldm}\}$, $\mathcal{R}_{\Box}^{\mathsf{x}} wu$ *iff for all* ϕ, *if* $\phi \in u$, *then* $\Diamond\phi \in w$. *(ii) For all* $\mathsf{x} \in \{\mathsf{pres}, \mathsf{fut}, \mathsf{Ldm}\}$, $\mathcal{R}_{i}^{\mathsf{x}} wu$ *iff for all* ϕ, *if* $\phi \in u$, *then* $\langle i \rangle \phi \in w$.

Lemma 5 (Existence Lemma [3]). *(i) For any world* $w^j \in W^{\mathsf{Ldm}}$, *if* $\Diamond\phi \in w^j$, *then there exists a world* $u^j \in W^{\mathsf{Ldm}}$ *such that* $\mathcal{R}_{\Box}^{\mathsf{Ldm}} w^j u^j$ *and* $\phi \in u^j$. *(ii) For any world* $w^j \in W^{\mathsf{Ldm}}$, *if* $\langle i \rangle \phi \in w^j$, *then there exists a world* $u^j \in W^{\mathsf{Ldm}}$ *such that* $\mathcal{R}_{i}^{\mathsf{Ldm}} w^j u^j$ *and* $\phi \in u^j$.

Lemma 6. *The Canonical Model is a temporal Kripke STIT model.*

Proof. We prove that M^{Ldm} has all the properties of a Tstit model:

- By lemma 1, the Ldm consistent set $\{p\}$ can be extended to a Ldm-MCS, and therefore W^{pres} is non-empty. Since $\mathbb{N}$ is non-empty as well, $W^{\mathsf{pres}} \times \mathbb{N} = W^{\mathsf{Ldm}}$ is a non-empty set of worlds.
- We argue that $\mathcal{R}_{\Box}^{\mathsf{Ldm}}$ is an equivalence relation between worlds of W^{Ldm}, and omit the arguments for $\mathcal{R}_{i}^{\mathsf{Ldm}}$ and $\mathcal{R}_{Ag}^{\mathsf{Ldm}}$, which are similar. Suppose that $w^j \in W^{\mathsf{Ldm}}$. We have two cases to consider: (i) $j = 0$, and (ii) $j > 0$. **(i)** Standard canonical model arguments apply and $\mathcal{R}_{\Box}^{\mathsf{Ldm}}$ is an equivalence relation between all worlds of the form $w^0 \in W^{\mathsf{Ldm}}$ (See [3] for details). **(ii)** If we fix a $j > 0$, then $\mathcal{R}_{\Box}^{\mathsf{Ldm}}$ will be an equivalence relation for all worlds of the form $w^j \in W^{\mathsf{Ldm}}$ since the intersection of equivalence relations produces another equivalence relation. Last, since $\mathcal{R}_{\Box}^{\mathsf{Ldm}}$ is an equivalence relation for each fixed $j \in \mathbb{N}$, and because each $W^{\mathsf{pres}} \times \{j\} \subset W^{\mathsf{Ldm}}$ is disjoint from each $W^{\mathsf{pres}} \times \{j'\} \subset W^{\mathsf{Ldm}}$ for $j \neq j'$, we know that the union all such equivalence relations will be an equivalence relation.

[1] Note that we choose to write each world $(w, j) \in W^{\mathsf{Ldm}}$ as w^j to simplify notation. Moreover, we write $\phi \in w^j$ to mean that the formula ϕ is in the Ldm-MCS w associated with j.

(C1) Let i be in Ag and assume that $(w^j, u^j) \in \mathcal{R}_i^{\mathsf{Ldm}}$. We split the proof into two cases: (i) $j = 0$, or (ii) $j > 0$. **(i)** Assume that $\Box\phi \in w^0$. Since w is a Ldm-MCS, it contains the axiom $\Box\phi \to [i]\phi$, and so, $[i]\phi \in w$ as well. Since $(w, u) \in \mathcal{R}_i^{\mathsf{pres}}$ (because $j = 0$), we know that $\phi \in u$ by the definition of the relation; therefore, $(w, u) \in \mathcal{R}_\Box^{\mathsf{pres}}$, which implies that $(w^0, u^0) \in \mathcal{R}_\Box^{\mathsf{Ldm}}$ by definition. **(ii)** The assumption that $j > 0$ implies that $u \in \mathcal{R}_i^{\mathsf{fut}}(w) = \bigcap_{i\in Ag} \mathcal{R}_i^{\mathsf{pres}}(w) = \mathcal{R}_\Box^{\mathsf{fut}}(w)$ by definition, which implies that $(w^j, u^j) \in \mathcal{R}_\Box^{\mathsf{Ldm}}$.

(C2) Let $u_1^j, ..., u_n^j \in W^{\mathsf{Ldm}}$ and assume that $\mathcal{R}_\Box^{\mathsf{Ldm}} u_i^j u_k^j$ for all $i, k \in \{1, ..., n\}$. We split the proof into two cases: (i) $j = 0$, or (ii) $j > 0$. **(i)** We want to show that there exists a world $w^j \in W^{\mathsf{Ldm}}$ such that $w^j \in \bigcap_{1\leq i\leq n} \mathcal{R}_i^{\mathsf{Ldm}}(u_i^j)$. Let $\hat{w}^j = \bigcup_{1\leq i\leq n}\{\phi | [i]\phi \in u_i^j\}$. Suppose that $\hat{w}^j$ is inconsistent to derive a contradiction. Then, there are ψ_1,...,ψ_k such that $\vdash_{\mathsf{Ldm}} \bigwedge_{1\leq l\leq k} \psi_i \to \bot$. For each $i \in Ag$, we define $\Phi_i = \{\psi_l | [i]\psi_l \in u_i^j\} \subseteq \{\psi_1, ..., \psi_k\}$. Observe that for each $i \in Ag$, $[i] \bigwedge \Phi_i \in u_i^j$ because $\bigwedge [i]\Phi_i \in u_i^j$ and $\vdash_{\mathsf{Ldm}} \bigwedge [i]\Phi_i \to [i] \bigwedge \Phi_i$. Since by assumption $\mathcal{R}_\Box^{\mathsf{Ldm}} u_i^j u_k^j$ for all $i, k \in \{1, ..., n\}$, this means that for any u_m^j we pick (with $1 \leq m \leq n$), $\Diamond[i] \bigwedge \Phi_i \in u_m^j$ for each $i \in Ag$ by lemma 4; hence, $\bigwedge_{i\in Ag} \Diamond[i] \bigwedge \Phi_i \in u_m^j$. By the ($\mathsf{IOA}$) axiom, this implies that $\Diamond \bigwedge_{i\in Ag}[i](\bigwedge \Phi_i) \in u_m^j$. By lemma 5, there must exist a world v^j such that $\mathcal{R}_\Box^{\mathsf{Ldm}} u_m^j v^j$ and $\bigwedge_{i\in Ag}[i](\bigwedge \Phi_i) \in v^j$. But then, since $\vdash_{\mathsf{Ldm}} [i](\bigwedge \Phi_i) \to \bigwedge \Phi_i$ by reflexivity, $\vdash_{\mathsf{Ldm}} \bigwedge_{i\in Ag}(\bigwedge \Phi_i) \leftrightarrow \bigwedge_{1\leq i\leq k} \psi_i$, and $\vdash_{\mathsf{Ldm}} \bigwedge_{1\leq i\leq k} \psi_i \to \bot$, it follows that $\bot \in v^j$, which is a contradiction since v^j is a Ldm-MCS. Therefore, $\hat{w}^j$ must be consistent and by lemma 1, it may be extended to a Ldm-MCS w^j. Since for each $[i]\phi \in u_i^j$, $\phi \in w^j$, we have that $w \in \mathcal{R}_i^{\mathsf{pres}}(u_i)$ for each $i \in Ag$. Hence, $w \in \bigcap_{1\leq i\leq n} \mathcal{R}_i^{\mathsf{pres}}(u_i)$, and so, $w^j \in \bigcap_{1\leq i\leq n} \mathcal{R}_i^{\mathsf{Ldm}}(u_i^j)$. **(ii)** Suppose that $j > 0$, so that $t^j \in \mathcal{R}_\Box^{\mathsf{Ldm}}(s^j)$ iff $t \in \mathcal{R}_\Box^{\mathsf{fut}}(s) = \bigcap_{i\in Ag} \mathcal{R}_i^{\mathsf{pres}}(s)$. By assumption then, $u_m^j \in \bigcap_{i\in Ag} \mathcal{R}_i^{\mathsf{pres}}(u_k^j) = \mathcal{R}_i^{\mathsf{fut}}(u_k^j)$ for all $k, m \in \{1, ..., n\}$ and each $i \in Ag$. Hence, $u_m^j \in \bigcap_{i\in Ag} \mathcal{R}_i^{\mathsf{fut}}(u_k^j)$ for all $k, m \in \{1, ..., n\}$. If we therefore pick any u_k^j, it follows that $u_k^j \in \bigcap_{i\in Ag} \mathcal{R}_i^{\mathsf{fut}}(u_i^j)$, meaning that the intersection $\bigcap_{1\leq i\leq n} \mathcal{R}_i^{\mathsf{Ldm}}(u_i^j)$ is non-empty.

(C3) Follows by definition.

– $\mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}}$ is a transitive and serial by definition, and $\mathcal{R}_{\mathsf{H}}^{\mathsf{Ldm}}$ is the converse of $\mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}}$ by definition as well.

(C4) For all $u^j, u^k, u^l \in W^{\mathsf{Ldm}}$, suppose that $\mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}} u^j u^k$ and $\mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}} u^j u^l$. Then, $j < k$ and $j < l$, and since $\mathbb{N}$ is linearly ordered, we have that $k < l$, $k = l$, or $k > l$, implying that $\mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}} u^k u^l$, $u^k = u^l$, or $\mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}} u^l u^k$.

(C5) Similar to previous case.

(C6) Suppose that $(u^j, v^{j+k}) \in \mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}} \circ \mathcal{R}_\Box^{\mathsf{Ldm}}$ with $k \geq 1$. By definition of $\mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}}$, u^{j+k} is the only element in $\mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}}(u^j)$ associated with $j+k$, and so, $(u^{j+k}, v^{j+k}) \in \mathcal{R}_\Box^{\mathsf{Ldm}}$ (By lemma 2 no other $u^{j+k'}$ with $k' \neq k$ can relate to v^{j+k} in $\mathcal{R}_\Box^{\mathsf{Ldm}}$.). Since $k \geq 1$, $v^{j+k} \in \mathcal{R}_\Box^{\mathsf{Ldm}}(u^{j+k})$ iff $v \in \mathcal{R}_\Box^{\mathsf{fut}}(u) = \bigcap_{i\in Ag} \mathcal{R}_i^{\mathsf{pres}}(u)$ iff $v^0 \in \mathcal{R}_{Ag}^{\mathsf{Ldm}}(u^0)$. By lemma 3, $(u^j, v^j) \in \mathcal{R}_{Ag}^{\mathsf{Ldm}}$. This implies that, and since $(v^j, v^{j+k}) \in \mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}}$ by definition, we have that $(u^j, v^{j+k}) \in \mathcal{R}_{Ag}^{\mathsf{Ldm}} \circ \mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}}$.

(C7) Follows from the definition of the $\mathcal{R}_{\mathsf{G}}^{\mathsf{Ldm}}$ relation.

- Last, it is easy to see that the valuation function V^{Ldm} is indeed a valuation function.

Lemma 7 (Truth-Lemma). *For any formula ϕ, $M^{\mathsf{Ldm}}, w^0 \models \phi$ iff $\phi \in w^0$.*

Proof. Shown by induction on the complexity of ϕ (See [3]).

B G3Xstit Derivation of IOA^x Axiom

We make use of the system of rules ($\mathsf{IOA_X}$), to derive the Xstit IOA axiom in G3Xstit.

$$\dfrac{\dfrac{\dfrac{\dfrac{R_\Box w_1w_2, R_\Box w_1w_3, R_\Box w_1w_4, R_A w_4w_5, R_A w_2w_5, w_2 : \langle A\rangle^x\overline{\phi}, w_3 : \langle B\rangle^x\overline{\psi}, \ldots \quad w_5 : \phi, w_5 : \overline{\phi}}{R_\Box w_1w_2, R_\Box w_1w_3, R_\Box w_1w_4, R_A w_4w_5, R_A w_2w_5, w_2 : \langle A\rangle^x\overline{\phi}, w_3 : \langle B\rangle^x\overline{\psi}, \ldots \quad w_5 : \phi}\ (\mathsf{IOA-U_1})}{R_\Box w_1w_2, R_\Box w_1w_3, R_\Box w_1w_4, R_A w_4w_5, w_2 : \langle A\rangle^x\overline{\phi}, w_3 : \langle B\rangle^x\overline{\psi}, \ldots \quad w_5 : \phi}}{R_\Box w_1w_2, R_\Box w_1w_3, R_\Box w_1w_4, w_2 : \langle A\rangle^x\overline{\phi}, w_3 : \langle B\rangle^x\overline{\psi}, \ldots \quad w_4 : [A]^x\phi}}{D_1}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{R_\Box w_1w_2, R_\Box w_1w_3, R_\Box w_1w_4, R_B w_4w_6, R_B w_3w_6, w_2 : \langle A\rangle^x\overline{\phi}, w_3 : \langle B\rangle^x\overline{\psi}, \ldots. \quad w_6 : \psi, w_6 : \overline{\psi}}{R_\Box w_1w_2, R_\Box w_1w_3, R_\Box w_1w_4, R_B w_4w_6, R_B w_3w_6, w_2 : \langle A\rangle^x\overline{\phi}, w_3 : \langle B\rangle^x\overline{\psi}, \ldots \quad w_6 : \psi}\ (\mathsf{IOA-U_2})}{R_\Box w_1w_2, R_\Box w_1w_3, R_\Box w_1w_4, R_B w_4w_6, w_2 : \langle A\rangle^x\overline{\phi}, w_3 : \langle B\rangle^x\overline{\psi}, \ldots \quad w_6 : \psi}}{R_\Box w_1w_2, R_\Box w_1w_3, R_\Box w_1w_4, w_2 : \langle A\rangle^x\overline{\phi}, w_3 : \langle B\rangle^x\overline{\psi}, \ldots \quad w_4 : [B]^x\psi}}{D_2}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{D_1 \qquad D_2}{R_\Box w_1w_2, R_\Box w_1w_3, R_\Box w_1w_4, w_2 : \langle A\rangle^x\overline{\phi}, w_3 : \langle B\rangle^x\overline{\psi}, w_1 : \Diamond([A]^x\phi \land [B]^x\psi), w_4 : [A]^x\phi \land [B]^x\psi}}{R_\Box w_1w_2, R_\Box w_1w_3, R_\Box w_1w_4, w_2 : \langle A\rangle^x\overline{\phi}, w_3 : \langle B\rangle^x\overline{\psi}, w_1 : \Diamond([A]^x\phi \land [B]^x\psi)}\ (\mathsf{IOA-E})}{R_\Box w_1w_2, R_\Box w_1w_3, w_2 : \langle A\rangle^x\overline{\phi}, w_3 : \langle B\rangle^x\overline{\psi}, w_1 : \Diamond([A]^x\phi \land [B]^x\psi)}}{w_1 : \Box\langle A\rangle^x\overline{\phi}, w_1 : \Box\langle B\rangle^x\overline{\psi}, w_1 : \Diamond([A]^x\phi \land [B]^x\psi)}}{w_1 : \Box\langle A\rangle^x\overline{\phi} \lor \Box\langle B\rangle^x\overline{\psi} \lor \Diamond([A]^x\phi \land [B]^x\psi)}$$

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